Basic convexity and Helly's Theorem: Worms on a line

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Segment and convex set

For two points $a, b \in \mathbb{R}^d$ we define a **segment** $[a, b]$ joining a and b as the set

$$[a, b] := \{\lambda a + (1 - \lambda)b : 0 \leq \lambda \leq 1\}.$$

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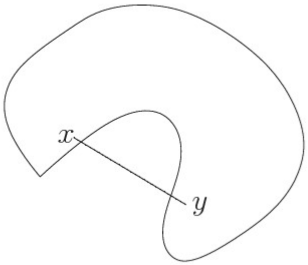
Examples of convex sets:

- The empty set $\emptyset$;
- An one-point set $\{a\}$, where $a \in \mathbb{R}^d$.
- For $a, b \in \mathbb{R}^d$ the segment $[a, b]$;
- $\mathbb{R}$;
- $\mathbb{R}^d$;
- The ball with center at $a \in \mathbb{R}^d$ and radius r: $\{x \in \mathbb{R}^d : \|x - a\| \leq r\}$.

Example of a non-convex set



Intersection of convex sets is convex

Fact (homework)

- For convex sets $A, B \subset \mathbb{R}^d$ their intersection $A \cap B$ is convex as well.
- For a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ their intersection $\bigcap \mathcal{F} = \bigcap_{K \in \mathcal{F}} K$ is convex as well.

Positive, affine, and convex combinations

For n points $a_1, \dots, a_n \in \mathbb{R}^d$ and n reals coefficients $\alpha_1, \dots, \alpha_n$, the point

$$a = \alpha_1 a_1 + \cdots + \alpha_n a_n$$

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- If the coefficients are positive, the linear combination a is called **a positive combination**.
- If the coefficients satisfy $\alpha_1 + \cdots + \alpha_n = 1$, the linear combination a is called **an affine combination**.
- If the coefficients satisfy both conditions $\alpha_1 + \cdots + \alpha_n = 1$ and $\alpha_i \geq 0$, the linear combination a is called **a convex combination**.

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Example. Let $a, b \in \mathbb{R}^d$. Every point of the segment $[a, b]$ is a convex combination of a and b with proper coefficients.

Positive, affine, and convex hulls

For a finite subset $S \subset \mathbb{R}^d$, define **the positive, affine, and convex hulls** as follows

$$\operatorname{span}_f S = \Big\{\text{all linear combinations of elements of } S\Big\};$$

$$\operatorname{co}_f S = \Big\{\text{all positive combinations of elements of } S\Big\},$$

$$\operatorname{aff}_f S = \Big\{\text{all affine combinations of elements of } S\Big\},$$

$$\operatorname{conv}_f S = \Big\{\text{all convex combinations of elements of } S\Big\}.$$

Usually we say 'linear hull' instead of '**span**'.
Also, people use the term '**cone**' instead of 'positive hull'.

Linear, positive, affine, and convex hulls

For any subset $S \subseteq \mathbb{R}^d$, define **the linear (span), positive (cone), affine, and convex hulls** as follows

$$\operatorname{span} S = \bigcup_{\text{finite } S' \subseteq S} \operatorname{span}_f S',$$

$$\operatorname{co} S = \bigcup_{\text{finite } S' \subseteq S} \operatorname{co}_f S',$$

$$\operatorname{aff} S = \bigcup_{\text{finite } S' \subseteq S} \operatorname{aff}_f S',$$

$$\operatorname{conv} S = \bigcup_{\text{finite } S' \subseteq S} \operatorname{conv}_f S'.$$

Linearly and affinely dependent vectors

We call vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ **linearly dependent** if there are reals $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ that are not all equal to zero such that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. As we know from Linear Algebra any $n > d$ vectors are linearly dependent.

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For a point $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, denote by $\begin{pmatrix} x \\ \alpha \end{pmatrix} \in \mathbb{R}^{d+1}$ the $(d+1)$-vector.

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Connection between linear and affine dependence

The vectors $v_1, \dots, v_n \in \mathbb{R}^d$ are linearly dependent iff the vectors $\begin{pmatrix} v_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} v_n \\ 1 \end{pmatrix}$ are affinely dependent.

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Any $n > d + 1$ vectors are affinely dependent.

Linear and affine independence

A set of vectors $v_1, \ldots, v_n$ is called **linear independent** if there are no reals $\lambda_1, \ldots, \lambda_n$ such that not all of them are equal to 0 and $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$.

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Convex and positive hulls

A point $z \in \mathbb{R}^d$ lies in the convex hull of $X \subseteq \mathbb{R}^d$ **iff**

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \sum_{i=1}^{n} \alpha_i \begin{pmatrix} x_i \\ 1 \end{pmatrix}$$

for some $x_1, \dots, x_n \in X$ and nonnegative α_i, that is, the point $\begin{pmatrix} z \\ 1 \end{pmatrix}$ lies in the cone (positive hull) of $\begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ 1 \end{pmatrix}$.

Helly's Theorem for worms

Helly's Theorem in $\mathbb{R}^1$

If any two segments of a finite family $\mathcal{F}$ of segments in $\mathbb{R}$ intersect, then all segments of $\mathcal{F}$ intersect.

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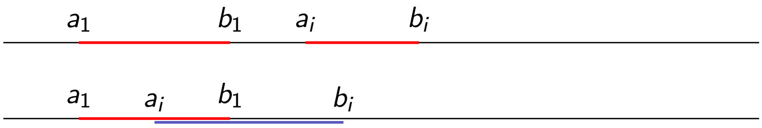
Since $[a_1, b_1] \cap [a_i, b_i] \neq \emptyset$, we have $b_1 \geq a_i$ because otherwise the segment $[a_1, b_1]$ lies on the left hand side with respect to $[a_i, b_i]$. Thus $b_1 \in [a_i, b_i]$. We are done.

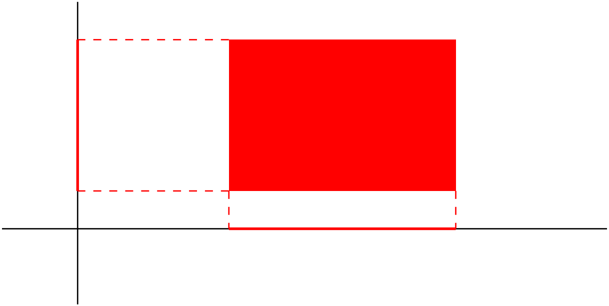


Helly's Theorem for rectangles

Recall that the Cartesian product $A \times B$ of two sets A and B is the set of ordered pairs (a, b) such that $a \in A$ and $b \in B$, that is,

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

For example, the Cartesian product of two segments $[a, b] \subset \mathbb{R}^1$ and $[c, d] \subset \mathbb{R}^1$ can be considered as a rectangle in $\mathbb{R}^2$ with sides parallel to the axis.



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If any two rectangles in a finite family $\mathcal{F}$ of rectangles in $\mathbb{R}^2$ with sides parallel to the axis intersect, then all rectangles of $\mathcal{F}$ intersect.

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Analogously replacing each rectangle $[a, b] \times [c, d]$ by a segment $[c, d]$, we obtain a family of segments on a line pairwise intersecting, and thus sharing a common point y.

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Analogously replacing each rectangle $[a, b] \times [c, d]$ by a segment $[c, d]$, we obtain a family of segments on a line pairwise intersecting, and thus sharing a common point y. Clearly, the point (x, y) belongs to all rectangles.

Helly's Theorem for parallelepipeds

Recall that the Cartesian product $A_1 \times \cdots \times A_d$ of d sets $A_1, \ldots, A_d$ is the set of d-tuples $(a_1, \ldots, a_d)$ such that $a_i \in A_i$, that is,

$$A_1 \times \cdots \times A_d = \{(a_1, \ldots, a_d) : a_i \in A_i \text{ for } 1 \leq i \leq d\}.$$

For example, the Cartesian product of d segments $[a_1, b_1], \ldots, [a_d, b_d] \subset \mathbb{R}^1$ can be considered as a d-dimensional parallelepiped with sides parallel to the axis.

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Helly's Theorem for parallelepipeds

If any two rectangles in a finite family $\mathcal{F}$ of parallelepipeds in $\mathbb{R}^d$ with sides parallel to the axis intersect, then all parallelepipeds of $\mathcal{F}$ intersect.

Colorful Helly's Theorem for worms

Colorful Helly's Theorem in $\mathbb{R}^1$

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be finite families of segments in $\mathbb{R}^1$ such that any two segments $s_1 \in \mathcal{F}_1$ and $s_2 \in \mathcal{F}_2$ intersect. Then there is $i \in \{1, 2\}$ such that all segments of the family $\mathcal{F}_i$ share a common point.

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Proof: Suppose that the segments of the family $\mathcal{F}_1$ do not have a common point. Then there are two disjoint segments $[a, b]$ and $[c, d]$ of the family $\mathcal{F}_1$. WLOG $b \leq c$. Take any segment $[x, y]$ of $\mathcal{F}_2$. Since it intersects with $[a, b]$, we have $x \leq b$. Analogously, since it intersects with $[c, d]$, we have $y \geq c$. Hence $[b, c] \subseteq [x, y]$ for any $[x, y] \in \mathcal{F}_1$. We are done.