The Radon and Carathéodory Theorems

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For example,

$$\big\{\{2n : n \in \mathbb{Z}\}, \{2n+1 : n \in \mathbb{Z}\}\big\} \text{ is a 2-partition of } \mathbb{Z},$$
$$\big\{\{1\}, \{2,5\}, \{3,4,6\}\big\} \text{ is a 3-partition of } [6]$$
$$\text{and is not a partition of } [7];$$
$$\big\{\{1,2\}, \{2,3\}\big\} \text{ is not a partition of } [3]$$
$$\big\{\emptyset, \{1\}, \{2\}\big\}.\text{is not a 3-partition of } [2].$$

Radon Theorem

Radon Theorem in $\mathbb{R}^2$

Let $X = \{x_1, x_2, x_3, x_4\}$ be set of four points in the plane $\mathbb{R}^2$. Then there is a partition $\{X_1, X_2\}$ of X such that

$$\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset.$$

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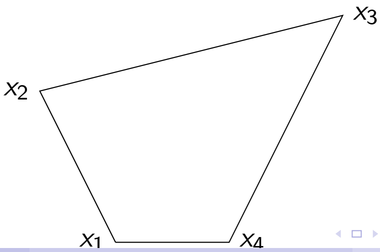
Proof. Consider the convex hull of X. There are several possibilities:

1. $\operatorname{conv} X$ is a quadrangle.



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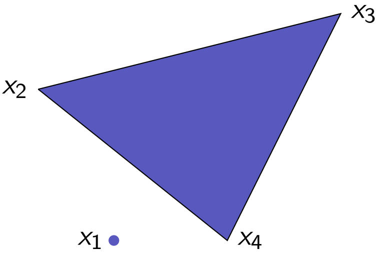

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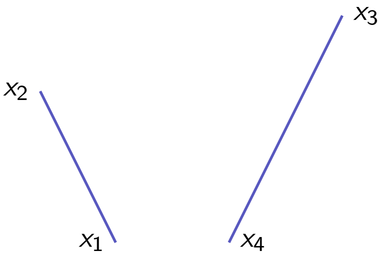

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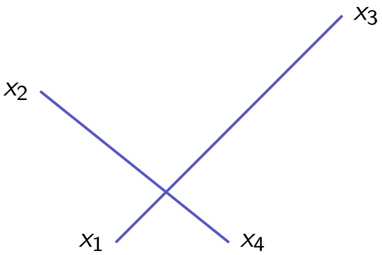

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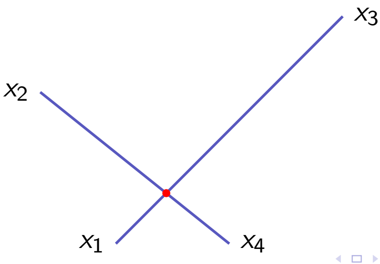
1. $\operatorname{conv} X$ is a quadrangle. **Yes,** $\{\{x_1, x_3\}, \{x_2, x_4\}\}$ **is the desired partition!**



Radon Theorem: Proof

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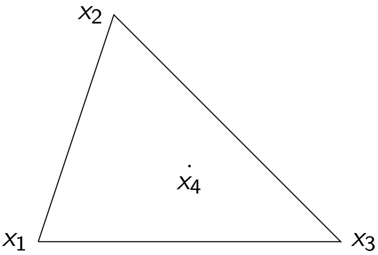

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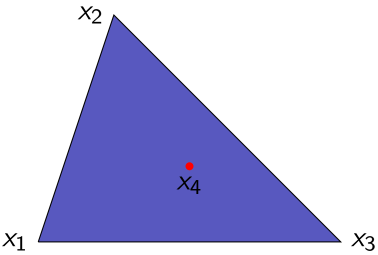

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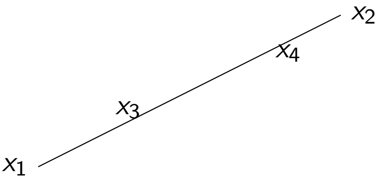

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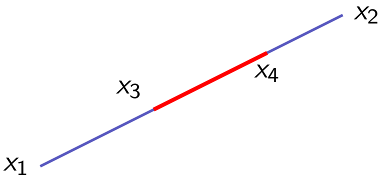

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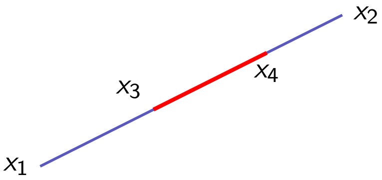

4. conv X is a point. $\{\{x_1, x_2\}, \{x_3, x_4\}\}$ **is the desired partition.**

$$x_1 = x_2 = x_3 = x_4$$

Radon Theorem

Radon Theorem: general case

If $X = \{x_1, x_2, \dots, x_{d+2}\}$ is a set of $d + 2$ points in $\mathbb{R}^d$. There is a 2-partition $\{X_1, X_2\}$ of X such

$$\operatorname{conv}(X_1) \cap \operatorname{conv}(X_2) \neq \emptyset.$$

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Proof. $d+2$ vectors $\binom{x_i}{1}$ in $\mathbb{R}^{d+1}$ are linearly dependent, that is,

$$\binom{0}{0} = \sum_{i=1}^{d+2} \alpha_i \binom{x_i}{1}$$

for some α_i not all of them are 0's.

Radon Theorem: proof

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \sum_{i=1}^{d+2} \alpha_i \begin{pmatrix} x_i \\ 1 \end{pmatrix}$$

Consider the partition $\big\{\{A_+, A_-\}\big\}$ of $[d+2]$ defined by

$$A_+ = \{i : \alpha_i > 0\} \text{ and } A_- = \{i : \alpha_i \leq 0\}.$$

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where $S > 0$. Therefore,

$$\sum_{i \in A_+} \frac{\alpha_i}{S} \begin{pmatrix} x_i \\ 1 \end{pmatrix} = \sum_{i \in A_-} \frac{(-\alpha_i)}{S} \begin{pmatrix} x_i \\ 1 \end{pmatrix} = \begin{pmatrix} y/S \\ 1 \end{pmatrix}.$$

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So, the point y/S belongs to $\operatorname{conv}\{x_i : i \in A_+\}$ and to $\operatorname{conv}\{x_i : i \in A_-\}$ at the same time. Thus, $\big\{\{x_i : i \in A_+\}, \{x_i : i \in A_-\}\big\}$ is the desired 2-partition.

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If $X \subset \mathbb{R}^d$ is a finite set of points with $\mathbf{z} \in \operatorname{conv} X$, then there exists a subset $Y \subseteq X$ with $|Y| \leq d + 1$ such that $z \in \operatorname{conv} Y$.

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The fact that z lies in the convex hull of X can be written as

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Proof of the Carathéodory Theorem

Thus these n vectors $\begin{pmatrix} x_i \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1}$ are linearly dependent, that is,

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