Basic convexity and the Helly theorem. Part 2

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Euclidean space

- We usually work with $\mathbb{R}^d = \{(x_1, \ldots, x_d) : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq d\}$, the space of all length d sequences $(x_1, \ldots, x_d)$.

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- Its elements are called vectors, the origin is $0 = (0, \ldots, 0)$.
- We add vectors coordinate-wise,

$$x + y = (x_1 + y_1, \ldots, x_d + y_d) \text{ for } x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d),$$

analogously we multiply by a scalar $\lambda \in \mathbb{R}$,

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- The standard basis is $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, 1)$, and so $x = \sum_{i=1}^{d} x_i e_i$.

Minkowski sum

- **The Minkowski sum** of two sets non-empty set $A, B \subseteq \mathbb{R}^d$ is defined by

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- The Minkowski sum of $A \subseteq R^d$ with a singleton $\{v\}$, where $v \in \mathbb{R}^d$, is called **a translation** of A and denoted as
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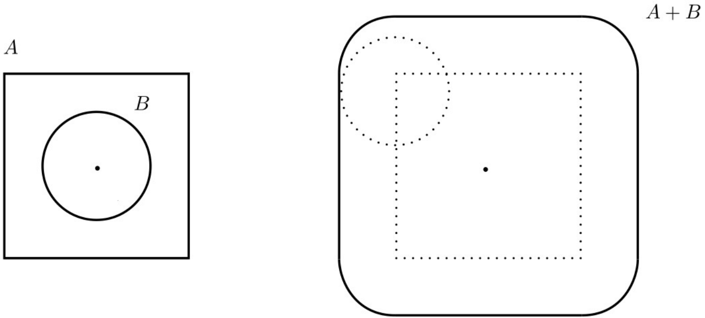
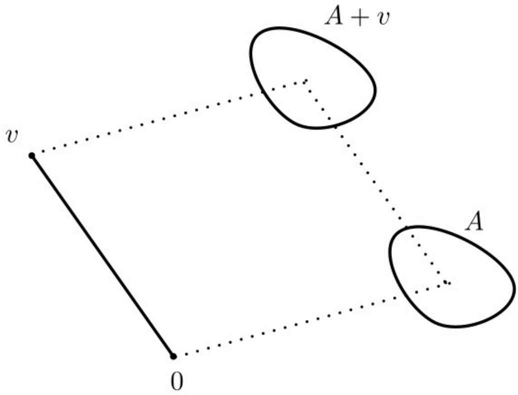
$$A + \{v\} = \{v\} + A = A + v = v + A.$$
- The set $v + \lambda A$ is called a **homothet** of A.

Minkowskii sum of a square and a disk



Translation by a vector



Questions

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Homework exercise

Prove that if $A, B \subset \mathbb{R}^d$ are convex then $A + B$ is also convex.

Scalar product

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- Properties:

$$\langle x, x \rangle \geq 0; \quad \langle x, y \rangle = \langle y, x \rangle; \quad \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle;$$

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle; \quad \langle \lambda x, y \rangle = \langle x, \lambda y \rangle = \lambda \langle x, y \rangle$$

for $x, y, z \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$.

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- What can we say about $x \in \mathbb{R}^d$ if $\langle x, x \rangle = 0$?

Euclidean norm

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$$|\lambda x| = |\lambda| \cdot |x| \text{ for every } \lambda \in \mathbb{R} \text{ and } x \in \mathbb{R}^d;$$

$$|x + y| \leq |x| + |y| \text{ for every } x, y \in \mathbb{R}^d.$$

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$$|x + y| \leq |x| + |y| \text{ for every } x, y \in \mathbb{R}^d.$$

- The Euclidean distance between points (the endpoints of corresponding vectors) x and y in $\mathbb{R}^d$ is $d(x, y) = |x - y|$.

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- From now on, assume $y \neq 0$. For any $\lambda \in \mathbb{R}$, we have a trivial inequality

$$0 \le \langle x - \lambda y, x - \lambda y\rangle = \langle x, x\rangle - 2\lambda\langle x, y\rangle + \lambda^2\langle y, y\rangle = (1)$$

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This implies the desired inequality $\langle x, y\rangle^2 \le \langle x, x\rangle\langle y, y\rangle$.

Useful Inequalities

Cauchy–Schwarz Inequality

For any reals $x_1, \ldots, x_n, y_1, \ldots, y_n$ we have

$$(x_1 \cdot y_1 + \cdots + x_n \cdot y_n)^2 \leq (x_1^2 + \cdots + x_n^2) \cdot (y_1^2 + \cdots + y_n^2).$$

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A function $f : [a, b] \to \mathbb{R}$ is called **convex** if

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda \cdot f(x) + (1-\lambda) \cdot f(y)$$

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Jensen's Inequality

For a convex function $f : [a, b] \to \mathbb{R}$, points $x_1, \dots, x_n \in [a, b]$, and positive reals $\lambda_1, \dots, \lambda_n$ with $\lambda_1 + \dots + \lambda_n = 1$, we have

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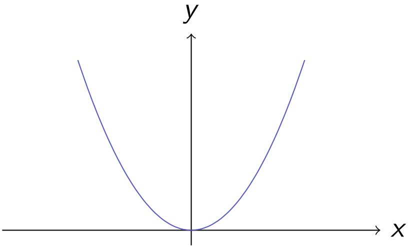
Consider the convex function $f(x) = x^2$.



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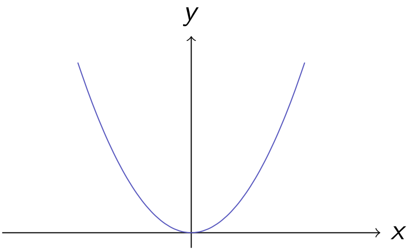

How to check convexity? Consider the second derivative $f''(x)$. If it is nonnegative at every point, then the function is convex.

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Consider the convex function $f(x) = x^2$. Then for any $x_1, \ldots, x_n$ we have

$$\left(\frac{x_1 + \cdots + x_n}{n}\right)^2 \leq \frac{1}{n}x_1^2 + \cdots + \frac{1}{n}x_n^2 \iff \frac{(x_1 + \cdots + x_n)^2}{n} \leq x_1^2 + \cdots + x_n^2.$$

Useful Inequalities

A function $f : [a, b] \to \mathbb{R}^d$ is called **concave** if

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

for any $0 \leq \lambda \leq 1$ and $x, y \in [a, b]$. If a function f is concave, then $-f$ is convex

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Jensen's inequality for concave functions

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If the second derivative $f''(x)$ is nonpositive at every point, then the function is concave.

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Consider a concave function $f(x) = \log x$ and positive points $x_1, \ldots, x_n$. Then

$$\log\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{1}{n}\log x_1 + \cdots + \frac{1}{n}\log x_n \iff$$

$$\log\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \log(x_1 \ldots x_n)^{1/n} \iff$$

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \ldots x_n}, \quad \text{the AM-GM Inequality}$$

Helly's theorem

Helly's theorem (1924)

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ such that any $d+1$ (or less) of them share a common point. Then the intersection of all convex sets of $\mathcal{F}$ is not empty.

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Corollary

Suppose that a finite set $\mathcal{P}$ of points in $\mathbb{R}^d$ is such that any $d+1$ of them can be covered by a ball of radius x. Then all points of $\mathcal{P}$ can be covered by a ball of radius x.

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Consider a family of balls of radius x with centers at elements of $\mathcal{F}$. Any $d+1$ of them share a common point, that is, satisfies the Helly Theorem. Thus, all balls have a common point.

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To prove the planar case, it is enough to show the following fact

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If $|a-b|, |b-c|, |c-a| \leq 1$, then the triangle abc can be covered by the disk of radius $1/\sqrt{3}$.

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$$\left\langle \frac{a+c}{2}, \frac{a+c}{2} \right\rangle \le \left\langle \frac{a-c}{2}, \frac{a-c}{2} \right\rangle \iff \langle a+c, a+c\rangle \le \langle a-c, a-c\rangle \iff$$

$$\langle a, a\rangle + 2\langle a, c\rangle + \langle c, c\rangle \le \langle a, a\rangle - 2\langle a, c\rangle + \langle c, c\rangle \iff \langle a, c\rangle \le 0.$$

Therefore, the disk with center at $\frac{a+c}{2}$ and radius $\frac{|a-c|}{2} \le \frac{1}{2} \le \frac{1}{\sqrt{3}}$ covers points $a, b = 0, c$.

Proof of the planar case

- Suppose the triangle abc is acute. It is well-known that the circumscribecenter o of abc lies inside of the triangle, that is, it lies in the convex hull of a, b and c. WLOG assume that o coinsides with the origin 0. Then

$$0 = \lambda_a a + \lambda_b b + \lambda_c c, \text{ where } \lambda_a + \lambda_b + \lambda_c = 1 \text{ and } \lambda_a, \lambda_b, \lambda_c \geq 0.$$

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$$0 = \lambda_a a + \lambda_b b + \lambda_c c, \text{ where } \lambda_a + \lambda_b + \lambda_c = 1 \text{ and } \lambda_a, \lambda_b, \lambda_c \geq 0.$$

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Next,

$$0 = \langle \lambda_a a + \lambda_b b + \lambda_c c, a\rangle = \lambda_a\langle a, a\rangle + \lambda_b\langle b, a\rangle + \lambda_c\langle c, a\rangle$$

$$\geq \lambda_a r^2 + \lambda_b(r^2 - 1/2) + \lambda_c(r^2 - 1/2) = (\lambda_a + \lambda_b + \lambda_c)r^2 - 1/2(\lambda_b + \lambda_c)$$

$$= r^2 - 1/2(1 - \lambda_a) \geq r^2 - 1/2 \cdot 2/3 = r^2 - 1/3, \text{ we are done.}$$

What to read?

- Basic Convexity and Helly. A. Barvinok, *A course in convexity,* Vol. 54. American Mathematical Soc., 2002. Sections 1.1, 1.2, 1.4.
- Inequalities. J. Michael Steel, *The Cauchy–Schwartz Inequality Master Class: an introduction to the art of mathematical inequalities*, Cambridge University Press, 2004. Chapters 1, 2, 6.