Helly's and Jung's theorems

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Helly Theorem

Helly Theorem: one-dimensional version

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Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ such that any $d+1$ (or less) of them share a common point. Then the intersection of all convex sets of $\mathcal{F}$ is not empty.

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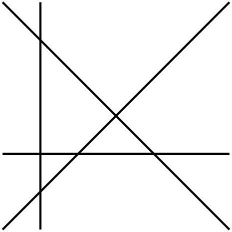
Question. Is it possible to replace $d+1$ by a smaller number? Say, by d. No! Here is an example in the plane.



Proof of the special case

Helly Theorem for 4 planar convex sets

Let C_1, C_2, C_3, C_4 be convex sets in the plane such that any three of them share a common point. Then $C_1 \cap C_2 \cap C_3 \cap C_4$ is not empty.

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$$x_3 \in C_1 \cap C_2 \cap C_4, x_4 \in C_1 \cap C_2 \cap C_3.$$

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Therefore,

$$x_2, x_3, x_4 \in C_1,\ x_1, x_3, x_4 \in C_2,\ x_1, x_2, x_4 \in C_3,\ x_1, x_2, x_3 \in C_4.$$

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Radon Theorem for the plane

Let x_1, x_2, x_3, x_4 be four points in the plane $\mathbb{R}^2$. Then there is a 2-partition $\{I_1, I_2\}$ of $\{1, 2, 3, 4\}$ such that $\text{conv}\{x_i : i \in I_1\} \cap \text{conv}\{x_i : i \in I_2\} \neq \emptyset$.

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Let us show that in any case a point

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There are two possible cases:

1. Both sets I_1 and I_2 have size 2.
2. One of the sets has size 1 and another has size 3.

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1. WLOG $I_1 = \{1, 2\}$ and $I_2 = \{3, 4\}$.

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We know that $x_1, x_2 \in C_3$, and thus, $[x_1, x_2] = \text{conv}\{x_1, x_2\} \subseteq C_3$. Since $y \in \text{conv}\{x_1, x_2\}$, we get $y \in C_3$.

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We know that $x_3, x_4 \in C_1$, and thus, $[x_3, x_4] = \operatorname{conv}\{x_3, x_4\} \subseteq C_1$. Since $y \in \operatorname{conv}\{x_3, x_4\}$, we get $y \in C_1$.

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We know that $x_1 \in C_2, C_3, C_4$. Let us show that $x_1 \in C_1$. Indeed, $\text{conv}\{x_1\} = \{x_1\}$, and thus, $x_1 \in \text{conv}\{x_2, x_3, x_4\}$. Since $x_2, x_3, x_4 \in C_1$, we get that $\text{conv}\{x_2, x_3, x_4\} \subseteq C_1$, which finishes the proof.

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By induction.
Base induction. If $|\mathcal{F}| \leq d+1$, then the statement is obvious.
Induction step. Suppose the statement is true for any family $\mathcal{F}$ of size at most n, where $n \geq d+1$.

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Let us show that the statement of the theorem holds for a family $\mathcal{F} = \{C_1, \ldots, C_{n+1}\}$. We know that for any $i \in [n+1]$, we have

$$\bigcap_{j \in [n+1] \setminus \{i\}} C_i \neq \emptyset \text{ (by the induction hypothesis).}$$

Proof

Choose a point

$$y_i \in \bigcap_{j\in[n+1]\setminus\{i\}} C_j.$$

- That is, $y_i \in C_j$ for $j \in [n+1] \setminus \{i\}$.

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- That is, $y_i \in C_j$ for $j \in [n+1] \setminus \{i\}$.
- For any $j \in [n+1]$, all points $y_1, \dots, y_{n+1}$ but y_j belong to C_j.

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- For any $j \in [n+1]$, all points $y_1, \ldots, y_{n+1}$ but y_j belong to C_j.
- Since C_j is convex, $\operatorname{conv}\{y_1, \ldots, y_{n+1}\} \setminus \{y_j\} \subseteq C_j$.

Proof

Radon Theorem

Let $x_1, x_2, \ldots, x_{d+2}$ be points in $\mathbb{R}^d$. Then there is a 2-partition $\{I_1, I_2\}$ of $[d+2]$ such that

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Radon Theorem for more points

Let $x_1, x_2, \dots, x_m$, where $m \geq d+2$, be points in $\mathbb{R}^d$. Then there is a 2-partition $\{I_1, I_2\}$ of $[m]$such that

$$\operatorname{conv}\{x_i : i \in I_1\} \cap \operatorname{conv}\{x_i : i \in I_2\} \neq \emptyset.$$

Since $n+1 \geq d+2$, we can apply the Radon Theorem for $y_1, \dots, y_{n+1}$.

Proof

Consider a 2-partition $\{I_1, I_2\}$ of $[n+1]$ that we obtain by the Radon Theorem. Choose any point

$$y \in \operatorname{conv}\{y_i : i \in I_1\} \cap \operatorname{conv}\{y_i : i \in I_2\} \neq \emptyset.$$

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- WLOG assume that $j \in I_1$ and $j \notin I_2$.
- Then all points y_i, $i \in I_2$, belong to C_j.
- Thus, $\operatorname{conv}\{y_i : i \in I_2\} \subset C_j$.
- Since $y \in \operatorname{conv}\{y_i : i \in I_2\}$, we are done.

Basic topology

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- A map $f : V \to \mathbb{R}^n$, where $V \subset \mathbb{R}^d$ is called **continuous** if for any point $x \in V$ and any $\varepsilon > 0$, there is $\delta > 0$ such that for any $y \in B(x, \delta)$ we have $f(y) \in B(f(x), \varepsilon)$.

Compactness

Theorem

Let $S \subseteq \mathbb{R}^d$. The following are equivalent:

- Any sequence of points of S has a subsequence converging to a point in S;
- S is closed and bounded.
- Any open cover of S has a finite subcover, that is, for a family $\mathcal{F}$ of open sets with
$$S \subseteq \bigcup_{A \in \mathcal{F}} A,$$
there is a finite subfamily $\mathcal{F}' \subseteq \mathcal{F}$ with
$$S \subseteq \bigcup_{A \in \mathcal{F}'} A.$$

If a set satisfies any of these properties, it is called **compact**.

Homework problem

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Let $\mathcal{F}$ be a (probably infinite) family of compact sets in $\mathbb{R}^d$. Let the intersection of elements of $\mathcal{F}$ be empty. Prove that there is a finite subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that the intersection of elements of $\mathcal{F}'$ is also empty.

Properties of compact

Theorem

1. Let $V \subset \mathbb{R}^d$ be a compact set and $f : V \to \mathbb{R}^n$ be a continuous map. Hence the image $f(V)$ is also compact.
2. Let $V \subset \mathbb{R}^d$ be a compact set and $f : V \to \mathbb{R}$ be a continuous function. Hence the function f attains its maximum (and minimum), that is, there is $x' \in V$ such that $f(x') \geq f(x)$ for any $x \in V$ (there is x'' such that $f(x'') \leq f(x)$ for any $x \in V$).

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Corollary of the Helly theorem

Suppose that a finite set $\mathcal{P}$ of points in $\mathbb{R}^d$ is such that any $d+1$ of them can be covered by a ball of radius x. Then all points of $\mathcal{P}$ can be covered by a ball of radius x.

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It is enough to show that...

Let $x_1, \ldots, x_{d+1}$ be points in $\mathbb{R}^d$ such that $|x_i - x_j| \leq 1$ for any i, j. Then there is a point o such that $|o - x_i|^2 \leq d/(2d+2)$.

Proof of the fact

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First, consider the function $f : \operatorname{conv}\{x_1, \ldots, x_{d+1}\} \to \mathbb{R}$ defined by

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Consider the two-partition X_0, X_1 of the set $X = \{x_1, \ldots, x_{d+1}\}$ define as:

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Consider the two-partition X_0, X_1 of the set $X = \{x_1, \dots, x_{d+1}\}$ define as: A point $x' \in X$ belongs to X_0 iff $|y - x'| = f(y)$, that is, a point $x'' \in X$ belongs to X_1 iff $|y - x''| < f(y)$.

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For the function $f : \text{conv}\{x_1, \ldots, x_{d+1}\} \to \mathbb{R}$ defined by

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Lemma (w/o proof)

$y \in \text{conv}\, X_0 = \text{conv}\{x_1, \ldots, x_k\}$.

That is, there are non-negative λ_i with $\lambda_1 + \cdots + \lambda_k = 1$ such that $\lambda_1 x_1 + \cdots + \lambda_k x_k = y$.

End of the proof

WLOG assume that $y = 0$ and $\lambda_1 \geq \lambda_i$ for $1 \leq i \leq k$. Thus $\lambda_1 \geq 1/k \geq 1/(d+1)$.

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- $1 \geq |x_1 - x_i|^2 = \langle x_1 - x_i, x_1 - x_i \rangle = 2r^2 - 2\langle x_1, x_i \rangle$ for $2 \leq i \leq k$, and thus, $\langle x_1, x_i \rangle \geq r^2 - 1/2$.

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$$0 = \langle x_1, \sum_{i=1}^{k} \lambda_i x_i \rangle = \lambda_1 \langle x_1, x_1 \rangle + \sum_{i=2}^{k} \lambda_i \langle x_1, x_i \rangle \geq$$

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$$r^2 - \frac{1 - \lambda_1}{2} \geq r^2 - \frac{1}{2} \Big(1 - \frac{1}{d+1} \Big)$$

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And thus $r^2 \leq d/(2d+2)$. We are done!