Colorful Caratheodory theorem

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- For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, we have $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- For any $\mathbf{u}, \mathbf{v} \in V$, we have $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- There is an element $\mathbf{0} \in V$ such that for any $\mathbf{v} \in V$, we have $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- For any $\mathbf{v} \in V$, there exists $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. We denote $\mathbf{w}$ as $-\mathbf{v}$.

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- For any $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v} \in V$, we have $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$
- For any vector $\mathbf{v}$, we have $1 \cdot \mathbf{v} = \mathbf{v}$
- For any $\alpha \in \mathbb{R}$ and $\mathbf{v}, \mathbf{u} \in V$, we have $\alpha(\mathbf{v} + \mathbf{u}) = \alpha\mathbf{v} + \alpha\mathbf{u}$
- For any $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v} \in V$, we have $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$

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Example of a vector subspace: The line

$$\{(x, y) \in \mathbb{R}^2 : x + 3y = 0\}$$

is a vector subspace of the plane $\mathbb{R}^2$.

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A set of vectors $\{\mathbf{v}_1, \dots, v_k\} \subset V$ is called **independent** if for $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ we have

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The **dimension** of a vector space V is the maximum size of its independent set. Without loss of generality we assume that the dimension is always finite. **Notation:** $\dim V$.

Affine subspace and hyperplane

Let $W \subseteq V$ be a vector subspace of V. Any translate of W, that is, the set $U = W + \mathbf{v}$ for some $\mathbf{v} \in V$, is called an **affine subspace** of V. The **dimension** of the affine subspace U is defined as the dimension of the vector subspace W. **Notation:** $\dim U = \dim(W + \mathbf{v})$.

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A vector subspace of the space $\mathbb{R}^d$ is called a **hyperplane** if its dimension is $d - 1$. A one-dimensional vector subspace is called a **line**. Usually we work with affine subspaces of $\mathbb{R}^d$ and call them **planes** or even **k-planes**.

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Question: What is a 0-dimensional plane?

Affine hull

The **affine hull** of a finite set $X = \{x_1, \dots, x_k\} \subset V$ is defined as

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$$\operatorname{aff} X = \bigcup_{Y \subseteq X : Y \text{ is finite}} \operatorname{aff}_f Y.$$

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A set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^d$ is called **affinely independent** if $\dim \operatorname{aff} X = k - 1$.

Hyperplane

Hyperplane (another definition)

A set X of points in $\mathbb{R}^d$ is called a **hyperplane** if there exists $\lambda \in \mathbb{R}$ and a non-zero vector $\mathbf{y} \in \mathbb{R}^d$ such that

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For example, a dot is a hyperplane in $\mathbb{R}^1$, a line is a hyperplane in $\mathbb{R}^2$; a plane is a hyperplane in $\mathbb{R}^3$.

Half-space

Open half-space

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The hyper-plane $\{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle = \lambda\}$ is called bounding for the open half-space $\{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle > \lambda\}$.

Comment

Clearly, the set $\{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle < \lambda\}$ is also an open half-space because

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Caratheodory Theorem: Classical and Colorful

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If $X \subset \mathbb{R}^d$ is a finite set of points such that $0 \in \operatorname{conv} X$, then there exists a subset $Y \subseteq X$ of size at most $d+1$ such that $0 \in \operatorname{conv} Y$.

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Colorful Caratheodory Theorem (Imre Bárány' 1982)

If $X_1, \dots, X_{d+1} \subset \mathbb{R}^d$ are finite sets of points such that for every $i \in [d+1]$ we have $0 \in \operatorname{conv} X_i$, then there exists $\mathbf{x}_i \in X_i$ for every $i \in [d+1]$ such that $0 \in \operatorname{conv}\{\mathbf{x}_1, \dots, \mathbf{x}_{d+1}\}$.

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Colorful implies Classical. Assume that $X \subset \mathbb{R}^d$ and 0 satisfy conditions of the Classical Caratheodory Theorem. Denoting $X_i = X$ for $i \in [d+1]$, we are in conditions of the Colorful Caratheodory Theorem.

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Sylvestor–Gallai theorem

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Proof. Suppose the contrary: all points of $\mathcal{P}$ do not lie on a line and for any two distinct points $a, b \in \mathcal{P}$ there is another point $c \in \mathcal{P}$ lying on a line passing through a and b.

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- Consider the set $\mathcal{L}$ of all possible lines passing through at least two points of X. Since X is finite, then the set $\mathcal{L}$ is finite as well.
- Since all points of X do not lie on a line, there is at least one pair $(a, \ell) \in \mathcal{P} \times \mathcal{L}$ such that $a \in \ell$, that is, $\text{dist}(a, \ell) > 0$.

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- Since $\mathcal{P}$ and $\mathcal{L}$ are finite, the set $\mathcal{P} \times \mathcal{L}$ is also finite. Therefore, there is a pair $(a, \ell) \in \mathcal{P} \times \mathcal{L}$ with the smallest positive distance between a and ℓ.

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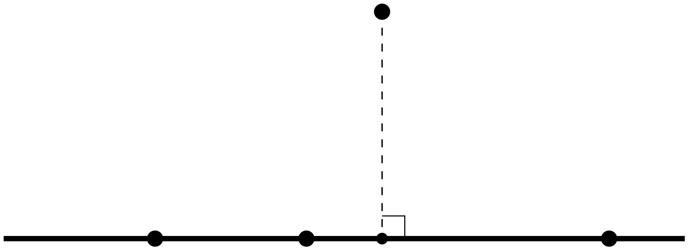

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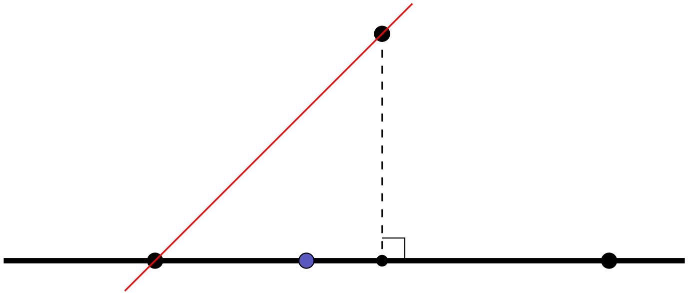

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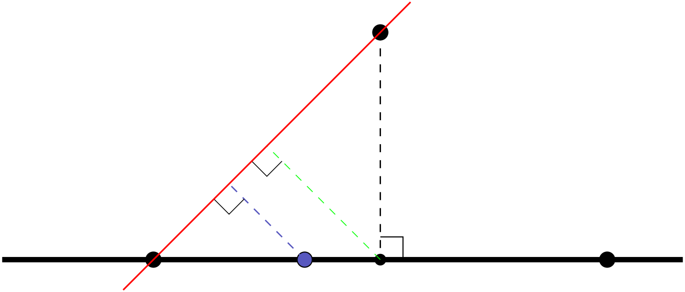

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Proof. We call a set $\{x_1, \ldots, x_{d+1}\}$ **colorful** if $x_i \in X_i$. Denote by $\mathcal{X}$ the set of all colorful sets. Since each X_i is finite, the set $\mathcal{X}$ is also finite. Suppose the contrary $0 \notin \operatorname{conv} X$ for all $X \in \mathcal{X}$, that is, $\operatorname{dist}(0, \operatorname{conv} X) > 0$.

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Reminder. The distance $\operatorname{dist}(X, Y)$ between sets $X, Y \subseteq \mathbb{R}^d$ defined as $\inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x} \in X, \mathbf{Y}\}$.

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Consider the continuous function $f : \mathcal{X} \to \mathbb{R}$ definite as $f(X) = \operatorname{dist}(0, \operatorname{conv} X)$ for $X \in \mathcal{X}$.

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Therefore, there is a point $\mathbf{a}_X \in \operatorname{conv} X$ such that $\|\mathbf{a}_X - \mathbf{o}\| = \operatorname{dist}(o, \operatorname{conv} X)$.

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Since we have finitely many colorful sets $X \in \mathcal{X}$, there is a set X_0 with the closest to 0 point $\mathbf{a}_X$. Notice that $0 \neq \mathbf{a}_X$. Put $a = a_X$.

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Since we have finitely many colorful sets $X \in \mathcal{X}$, there is a set X_0 with the closest to 0 point $\mathbf{a}_X$. Notice that $0 \neq \mathbf{a}_X$. Put $a = a_X$.

Consider a hyperplane H through $\mathbf{a}_{Z'}$ orthogonal to the segment $0\mathbf{a}_X$.

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Fact 1

$X_0 \subset H^+$

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Homework problem

Prove that if $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are distinct points of $\mathbb{R}^d$ such that $\angle \mathbf{xyz} < \pi/2$, then there is $\mathbf{t}$ of the open segment $(\mathbf{z}, \mathbf{y})$ with $|\mathbf{x} - \mathbf{t}| < |\mathbf{x} - \mathbf{y}|$.

Colorful Caratheodory Theorem: Proof

Fact 2

$a \in \mathrm{conv}(X_0 \cap H)$

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Let $H_+ \subseteq \mathbb{R}^d$ be a close half-space with the bounding hyperplane H and X be any subset of $\mathbb{R}^d$. Prove that if $X \subset H_+$, then $\text{conv}(X \cap H) = \text{conv}\, X \cap H$.

- By the Caratheodory theorem, the point a lies in the convex hull of at most d points of X. WLOG $a \in \text{conv}(X_0 \setminus x_1)$, where $x_1 \in X_0$ and $x_1 \in X_1$.

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- Since $0 \in \mathrm{conv}\, X_1$, there is a point $x_1' \in X_1$ lying in H^- because otherwise $X_1 \subset H^+$ and thus $\mathrm{conv}\, X_1 \subset H^+$ and it does not contain 0.

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End of the proof

Thus the convex hull of the colorful set $\{x_1'\} \cup X_0 \setminus \{x_1\}$ is closer to 0 then the convex hull of X_0.

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