Colorful Helly's theorem

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$$x_1 = y_1, \ldots, x_{i-1} = y_{i-1}, x_i < y_i.$$

Analogously we write **xy** if either $\mathbf{x} \leq_{lex} \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$.

Auxiliary Lemmas

Lemma 1

For a non-empty compact set $K \subset \mathbb{R}$ there is the point $\mathbf{x} \in K$ such that $\mathbf{y} <_{\mathrm{lex}} \mathbf{x}$ for any $\mathbf{y} \in K \setminus \{\mathbf{x}\}$.

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Lemma 1 (planar case)

For a non-empty compact set $K \subset \mathbb{R}^2$ there is the point $\mathbf{x} \in K$ such that $\mathbf{y} <_{\text{lex}} \mathbf{x}$ for any $\mathbf{y} \in K \setminus \{\mathbf{x}\}$.

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The point $\mathbf{x}$ is called **the lexicographic maximum** of K. **Notation**: (K).

Some observations

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Since $(K_1) \in K_1 \subseteq K_2$, we get $(K_1)(K_2)$.

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Lemma 2 (planar version)

Let $K_1, K_2, K_3 \subset \mathbb{R}^2$ be convex compact sets such that $K_1 \cap K_2 \cap K_3 \neq \emptyset$.

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Lemma 2 (planar version)

Let $K_1, K_2, K_3 \subset \mathbb{R}^2$ be convex compact sets such that $K_1 \cap K_2 \cap K_3 \neq \emptyset$. Then the lexicographic maximum of $K_1 \cap K_2 \cap K_3$ is the lexicographic maximum of one of the following sets:

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Lemma 2 (general version)

Let $K_1, \ldots, K_{d+1} \subset \mathbb{R}^d$ be convex compact sets such that $K_1 \cap \cdots \cap K_{d+1} \neq \emptyset$. Then the lexicographic maximum of $K_1 \cap \cdots \cap K_3$ is the lexicographic maximum of one of the following sets:

$$\bigcap_{j \in [d+1] \setminus \{i\}} K_j \text{ for } 1 \leq i \leq d+1.$$

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Let x be the lexicographic maximum of $K_1 \cap K_2 \cap K_3$.

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- By Helly's theorem, there are 3 convex sets among K, K_1, K_2, K_3 with empty intersection.

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- Since $K_1 \cap K_2 \cap K_3 \neq \emptyset$, WLOG we may assume that $K \cap K_1 \cap K_2 = \emptyset$.
- Consider the lexicographic maximum x_{12} of $K_1 \cap K_2$.

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- By Helly's theorem, there are 3 convex sets among K, K_1, K_2, K_3 with empty intersection.
- Since $K_1 \cap K_2 \cap K_3 \neq \emptyset$, WLOG we may assume that $K \cap K_1 \cap K_2 = \emptyset$.
- Consider the lexicographic maximum x_{12} of $K_1 \cap K_2$.
- We know: xx_{12} and $x_{12} \notin K$. Thus $x = x_{12}$.

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- Choose a triple $(K_1, K_2, K_3) \in \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{F}_3$ such that
  $\max(K_1 \cap K_2 \cap K_3) \max(K_1' \cap K_2' \cap K_3')$
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- By Lemma 2, WLOG we assume $\max(K_1 \cap K_2) = \max(K_1 \cap K_2 \cap K_3) \max(K_1 \cap K_2 \cap K_3')$ for arbitrary $K_3' \in \mathcal{F}_3$.
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- Since $K_1 \cap K_2 \cap K_3' \subseteq K_1 \cap K_2$, we have $\max(K_1 \cap K_2) = \max(K_1 \cap K_2 \cap K_3')$ for arbitrary $K_3' \in \mathcal{F}_3$.