Tverberg's theorem

Alexander Polyanskii
http://polyanskii.com

Moscow Institute of Physics and Technology

Radon's and Tverberg's theorems

Radon's theorem (1921)

For a set $X \subset \mathbb{R}^d$ of size at least $d+2$, there is a 2-partition $\{X_1, X_2\}$ of X such that

$$\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset.$$

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Tverberg's theorem (1966)

Let $r \geq 2$ and $d \geq 1$ be integers. For a set $X \subseteq \mathbb{R}^d$ of size (at least) $(r-1)(d+1)+1$, there is a r-partition $\{X_1, \dots, X_r\}$ of X such that

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Radon's theorem corresponds to the case $r = 2$ of Tverberg's theorem.

Tverberg's theorem: Proof for $r = 3$

Denote points of X by $\mathbf{p}_1, \ldots, \mathbf{p}_{2d+3} \in \mathbb{R}^d$. Out goal is to find 3-partition $\{A_1, A_2, A_3\}$ of $[2d+3]$ such that

$$\begin{bmatrix} \sum_{i \in A_1} \alpha_i \mathbf{p}_i \\ \sum_{i \in A_1} \alpha_i \end{bmatrix} = \sum_{i \in A_1} \alpha_i \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix} = \sum_{i \in A_2} \alpha_i \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix} = \sum_{i \in A_3} \alpha_i \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix} \in \mathbb{R}^{d+1},$$

where $\alpha_i \geq 0$ and $\sum_{i \in A_j} \alpha_i = 1$ for $j = 1, 2, 3$.

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where $\alpha_i \geq 0$ and $\sum_{i \in A_j} \alpha_i = 1$ for $j = 1, 2, 3$.
Replacing $\alpha_i \to 3\alpha_i$, we easily obtain the following conditions:

$$\sum_{i \in A_1} \alpha_i \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix} = \sum_{i \in A_2} \alpha_i \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix} = \sum_{i \in A_3} \alpha_i \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix},$$

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where $\alpha_i \geq 0$ and $\sum_{i=1}^{2d+3} \alpha_i = 1$. Denote

$$S_j := \sum_{i \in A_j} \alpha_j \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix}.$$

Tverberg's theorem: Proof for $r = 3$

$$S_1 = \sum_{i \in A_1} \alpha_i \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix} = S_2 = \sum_{i \in A_2} \alpha_i \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix} = S_3 = \sum_{i \in A_3} \alpha_i \begin{bmatrix} \mathbf{p}_i \\ 1 \end{bmatrix},$$

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where $\alpha_i \geq 0$ and $\sum_{i=1}^{2d+3} \alpha_i = 1$.
We can rewrite the above conditions as

$$S_1 = S_2 = S_3 \iff \begin{bmatrix} S_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ S_2 \end{bmatrix} + \begin{bmatrix} -S_3 \\ -S_3 \end{bmatrix} = 0.$$

Denoting

$$\varphi_{1,i} = \begin{bmatrix} \mathbf{p}_i \\ 1 \\ \mathbf{0} \end{bmatrix}, \quad \varphi_{2,i} = \begin{bmatrix} \mathbf{0} \\ \mathbf{p}_i \\ 1 \end{bmatrix}, \quad \varphi_{3,i} = \begin{bmatrix} -\mathbf{p}_i \\ -1 \\ -\mathbf{p}_i \\ -1 \end{bmatrix} \in \mathbb{R}^{2d+2},$$

we obtain

$$\sum_{i \in A_1} \alpha_i \varphi_{1,i} + \sum_{i \in A_2} \alpha_i \varphi_{2,i} + \sum_{i \in A_3} \alpha_i \varphi_{3,i} = \mathbf{0}.$$

Tverberg's theorem: Proof for $r = 3$

Colorful Caratheodory's theorem

If $X_1, \ldots, X_{n+1} \subset \mathbb{R}^n$ are finite sets of points in $\mathbb{R}^n$ such that $\mathbf{o} \in \operatorname{conv} X_i$, then there exists $\mathbf{x}_i \in X_i$ for every $i \in [n+1]$ such that $\mathbf{o} \in \operatorname{conv}\{\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$.

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Since $\mathbf{0} = \varphi_{1,i} + \varphi_{2,i} + \varphi_{3,i}$, we have $\mathbf{0} \in \operatorname{conv}\{\varphi_{1,i}, \varphi_{2,i}, \varphi_{3,i}\}$ for every $i \in [2d+3]$.

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Since $\mathbf{0} = \varphi_{1,i} + \varphi_{2,i} + \varphi_{3,i}$, we have $\mathbf{0} \in \operatorname{conv}\{\varphi_{1,i}, \varphi_{2,i}, \varphi_{3,i}\}$ for every $i \in [2d+3]$. Therefore, we can apply Colorful Caratheodory's theorem for $n = 2d+2$ and $X_i = \{\varphi_{1,i}, \varphi_{2,i}, \varphi_{3,i}\}$.

Tverberg's theorem: Proof for $r = 3$

Colorful Caratheodory's theorem

If $X_1, \ldots, X_{n+1} \subset \mathbb{R}^n$ are finite sets of points in $\mathbb{R}^n$ such that $\mathbf{o} \in \operatorname{conv} X_i$, then there exists $\mathbf{x}_i \in X_i$ for every $i \in [n+1]$ such that $\mathbf{o} \in \operatorname{conv}\{\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$.

Since $\mathbf{0} = \varphi_{1,i} + \varphi_{2,i} + \varphi_{3,i}$, we have $\mathbf{0} \in \operatorname{conv}\{\varphi_{1,i}, \varphi_{2,i}, \varphi_{3,i}\}$ for every $i \in [2d+3]$. Therefore, we can apply Colorful Caratheodory's theorem for $n = 2d+2$ and $X_i = \{\varphi_{1,i}, \varphi_{2,i}, \varphi_{3,i}\}$. Thus, $\mathbf{0} \in \operatorname{conv}\{x_1, \ldots, x_{2d+3}\}$, where $x_i \in X_i$.

Tverberg's theorem: Proof for $r = 3$

Thus, $\mathbf{0} \in \operatorname{conv}\{x_1, \dots, x_{2d+3}\}$, where $x_i \in X_i = \{\varphi_{1,i}, \varphi_{2,i}, \varphi_{3,i}\}$. Denote

$$A_j = \{i \in [2d+3] : x_i = \varphi_{j,i}\}.$$

Thus, $\mathbf{0} \in \operatorname{conv}\{x_1, \ldots, x_{2d+3}\}$, where $x_i \in X_i = \{\varphi_{1,i}, \varphi_{2,i}, \varphi_{3,i}\}$. Denote

$$A_j = \{i \in [2d+3] : x_i = \varphi_{j,i}\}.$$

Since $\mathbf{0} \in \operatorname{conv}\{\mathbf{x}_1, \ldots, \mathbf{x}_{2d+3}\}$, there are $\alpha_i \geq 0$ such that $\sum_{i=1}^{2d+3} \alpha_i = 1$ and

$$\sum_{i \in A_1} \alpha_i \varphi_{1,i} + \sum_{i \in A_2} \alpha_i \varphi_{2,i} + \sum_{i \in A_3} \alpha_i \varphi_{3,i} = 0$$

hold.