Separation theorems and polarity

Alexander Polyanskii
http://polyanskii.com

Moscow Institute of Physics and Technology

Separation theorem

A hyperplane h is called **strictly separating** two subsets sets A and B of $\mathbb{R}^d$ if they lie in different **open** half-spaces bounded by h.

Separation theorem

A hyperplane h is called **strictly separating** two subsets sets A and B of $\mathbb{R}^d$ if they lie in different **open** half-spaces bounded by h.

Separation theorem (weak version)

Let $A, B \subset \mathbb{R}^d$ be convex compact (non-empty) sets such that $A \cap B = \emptyset$.

Separation theorem

A hyperplane h is called **strictly separating** two subsets sets A and B of $\mathbb{R}^d$ if they lie in different **open** half-spaces bounded by h.

Separation theorem (weak version)

Let $A, B \subset \mathbb{R}^d$ be convex compact (non-empty) sets such that $A \cap B = \emptyset$. Then there is a hyperplane strictly separating A and B.

Separation theorem

A hyperplane h is called **strictly separating** two subsets sets A and B of $\mathbb{R}^d$ if they lie in different **open** half-spaces bounded by h.

Separation theorem (weak version)

Let $A, B \subset \mathbb{R}^d$ be convex compact (non-empty) sets such that $A \cap B = \emptyset$. Then there is a hyperplane strictly separating A and B.

Proof. Since $A \times B \subset \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$ is a compact set, the continius function $f : A \times B \to \mathbb{R}$ defined by $f(\mathbf{x} \times \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$

Separation theorem

A hyperplane h is called **strictly separating** two subsets sets A and B of $\mathbb{R}^d$ if they lie in different **open** half-spaces bounded by h.

Separation theorem (weak version)

Let $A, B \subset \mathbb{R}^d$ be convex compact (non-empty) sets such that $A \cap B = \emptyset$. Then there is a hyperplane strictly separating A and B.

Proof. Since $A \times B \subset \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$ is a compact set, the continius function $f : A \times B \to \mathbb{R}$ defined by $f(\mathbf{x} \times \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ attains its minimum at some point $\mathbf{a} \times \mathbf{b} \in A \times B$.

Separation theorem

A hyperplane h is called **strictly separating** two subsets sets A and B of $\mathbb{R}^d$ if they lie in different **open** half-spaces bounded by h.

Separation theorem (weak version)

Let $A, B \subset \mathbb{R}^d$ be convex compact (non-empty) sets such that $A \cap B = \emptyset$. Then there is a hyperplane strictly separating A and B.

Proof. Since $A \times B \subset \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$ is a compact set, the continius function $f : A \times B \to \mathbb{R}$ defined by $f(\mathbf{x} \times \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ attains its minimum at some point $\mathbf{a} \times \mathbf{b} \in A \times B$. Since $A \cap B = \emptyset$, we have $f(\mathbf{a} \times \mathbf{b}) > 0$.

Separation theorem

A hyperplane h is called **strictly separating** two subsets sets A and B of $\mathbb{R}^d$ if they lie in different **open** half-spaces bounded by h.

Separation theorem (weak version)

Let $A, B \subset \mathbb{R}^d$ be convex compact (non-empty) sets such that $A \cap B = \emptyset$. Then there is a hyperplane strictly separating A and B.

Proof. Since $A \times B \subset \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$ is a compact set, the continius function $f : A \times B \to \mathbb{R}$ defined by $f(\mathbf{x} \times \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ attains its minimum at some point $\mathbf{a} \times \mathbf{b} \in A \times B$. Since $A \cap B = \emptyset$, we have $f(\mathbf{a} \times \mathbf{b}) > 0$. Consider the perpendicular bisector h of $\mathbf{a}$ and $\mathbf{b}$. Let us show that is a desired hyperplane.

End of the proof

Recall the following lemma/problem.

Lemma

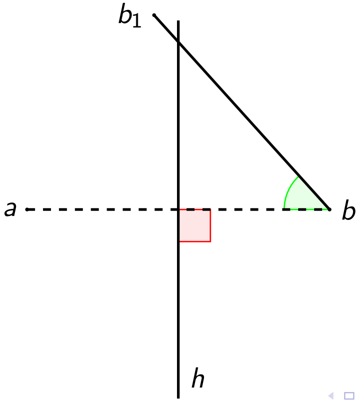
If $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are distinct points of $\mathbb{R}^d$ such that $\angle \mathbf{xyz} < \pi/2$, then there is $\mathbf{t}$ of the open segment $(\mathbf{z}, \mathbf{y})$ with $|\mathbf{x} - \mathbf{t}| < |\mathbf{x} - \mathbf{y}|$.

End of the proof

Recall the following lemma/problem.

Lemma

If $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are distinct points of $\mathbb{R}^d$ such that $\angle\mathbf{xyz} < \pi/2$, then there is $\mathbf{t}$ of the open segment $(\mathbf{z}, \mathbf{y})$ with $|\mathbf{x} - \mathbf{t}| < |\mathbf{x} - \mathbf{y}|$.



Isolation Theorem

A hyperplane h is called **separating** two sets $A, B \subset \mathbb{R}^d$ if they lie in different **closed** half-spaces bounded by h.

Isolation Theorem

A hyperplane h is called **separating** two sets $A, B \subset \mathbb{R}^d$ if they lie in different **closed** half-spaces bounded by h.

Isolation Theorem

If $\mathbf{a}$ does not belong to the convex set A, then there is a hyperplane passing through $\mathbf{a}$ and separating $\{\mathbf{a}\}$ and A.

Without a proof.

Isolation Theorem

A hyperplane h is called **separating** two sets $A, B \subset \mathbb{R}^d$ if they lie in different **closed** half-spaces bounded by h.

Isolation Theorem

If $\mathbf{a}$ does not belong to the convex set A, then there is a hyperplane passing through $\mathbf{a}$ and separating $\{\mathbf{a}\}$ and A.

Without a proof.

A **convex body** is a convex compact set with non-empty interior, that is, there is an open ball lying in the set.

Existence of supporting hyperplanes

Let $\mathbf{a} \in \partial K$ be a boundary point of a convex compact set K. Then there exists a hyperplane passing through $\mathbf{a}$ such that K lies in one of closed half-spaces formed by this hyperplane.

Existence of supporting hyperplane

Consider the interior of K, that is,

$$\operatorname{int} K = \{\mathbf{x} : \text{there exists a ball } B(\mathbf{x}, r) \subseteq K\} \neq \emptyset.$$

Homework problem. The set $\operatorname{int} K$ is convex.

Existence of supporting hyperplane

Consider the interior of K, that is,

$$\operatorname{int} K = \{\mathbf{x} : \text{there exists a ball } B(\mathbf{x}, r) \subseteq K\} \neq \emptyset.$$

Homework problem. The set $\operatorname{int} K$ is convex.

By the Isolation Theorem, for sets $\{\mathbf{a}\}$ and $\operatorname{int} K$ there exists a hyperplane passing through $\mathbf{a}$ such that $\operatorname{int} K$ lies in one of two closed half-spaces formed by it.

Existence of supporting hyperplane

Consider the interior of K, that is,

$$\operatorname{int} K = \{\mathbf{x} : \text{there exists a ball } B(\mathbf{x}, r) \subseteq K\} \neq \emptyset.$$

Homework problem. The set $\operatorname{int} K$ is convex.

By the Isolation Theorem, for sets $\{\mathbf{a}\}$ and $\operatorname{int} K$ there exists a hyperplane passing through $\mathbf{a}$ such that $\operatorname{int} K$ lies in one of two closed half-spaces formed by it.

Homework problem. Prove that K also lies in the closed halfspace.

Separation theorem

Strong Separation theorem (aka the Hahn–Banach theorem)

If A and B are convex set in $\mathbb{R}^d$ without common points, then there is a hyperplane separating A and B.

Proof. Consider the convex set $C = A + (-B)$. Clearly, the origin does not belong to C.

Separation theorem

Strong Separation theorem (aka the Hahn–Banach theorem)

If A and B are convex set in $\mathbb{R}^d$ without common points, then there is a hyperplane separating A and B.

Proof. Consider the convex set $C = A + (-B)$. Clearly, the origin does not belong to C. By the Isolation theorem, there is a hyperplane

$$\{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle = 0\}$$

separating $\{\mathbf{o}\}$ and C, where $\mathbf{y}$ is some non-zero vector.

Strong Separation theorem (aka the Hahn–Banach theorem)

If A and B are convex set in $\mathbb{R}^d$ without common points, then there is a hyperplane separating A and B.

Proof. Consider the convex set $C = A + (-B)$. Clearly, the origin does not belong to C. By the Isolation theorem, there is a hyperplane

$$\{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle = 0\}$$

separating $\{\mathbf{o}\}$ and C, where $\mathbf{y}$ is some non-zero vector. Without loss of generality assume that

$$C \subseteq \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \geq 0\}.$$

Hence for any $\mathbf{a} \in A$ and $\mathbf{b} \in B$, the following inequality holds

$$\langle \mathbf{a} - \mathbf{b}, \mathbf{y} \rangle \geq 0,$$

that is, $\langle \mathbf{a}, \mathbf{y} \rangle \geq \langle \mathbf{b}, \mathbf{y} \rangle$ for any $\mathbf{a} \in A$, $\mathbf{b} \in B$.

Proof of Separtion theorem

Therefore, the following inequality holds

$$\inf_{\mathbf{a}\in A}\langle \mathbf{a}, \mathbf{y}\rangle \geq \sup_{\mathbf{b}\in B}\langle \mathbf{b}, \mathbf{y}\rangle.$$

So, choosing any $\gamma \in [\sup_{\mathbf{b}\in B}\langle \mathbf{b}, \mathbf{y}\rangle, \inf_{\mathbf{a}\in A}\langle \mathbf{a}, \mathbf{y}\rangle]$, we obtain

$$\langle \mathbf{a}, \mathbf{y}\rangle \geq \gamma \geq \langle \mathbf{b}, \mathbf{y}\rangle.$$

So, the hyperplane $\{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y}\rangle = \gamma\}$ is desired.

Polarity

Let $A \subset \mathbb{R}^d$ be a non-empty set. The set

$$A^\circ = \bigcap_{\mathbf{x} \in A} \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1\}.$$

is called the polar of A.

Polarity

Let $A \subset \mathbb{R}^d$ be a non-empty set. The set

$$A^\circ = \bigcap_{\mathbf{x} \in A} \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1\}.$$

is called the polar of A.

Properties of the polar set

1. A° is a closed convex set in $\mathbb{R}^d$ and the origin lies in A°.

Polarity

Let $A \subset \mathbb{R}^d$ be a non-empty set. The set

$$A^\circ = \bigcap_{\mathbf{x} \in A} \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1\}.$$

is called the polar of A.

Properties of the polar set

1. A° is a closed convex set in $\mathbb{R}^d$ and the origin lies in A°.
2. If $A \subseteq B$ then $B^\circ \subseteq A^\circ$.

Polarity

Let $A \subset \mathbb{R}^d$ be a non-empty set. The set

$$A^\circ = \bigcap_{\mathbf{x} \in A} \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1\}.$$

is called the polar of A.

Properties of the polar set

1. A° is a closed convex set in $\mathbb{R}^d$ and the origin lies in A°.
2. If $A \subseteq B$ then $B^\circ \subseteq A^\circ$.
3. $\left(\cup_{i \in I} A_i \right)^\circ = \cap_{i \in I} A_i^\circ$.

Polarity

Let $A \subset \mathbb{R}^d$ be a non-empty set. The set

$$A^\circ = \bigcap_{\mathbf{x} \in A} \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1\}.$$

is called the polar of A.

Properties of the polar set

1. A° is a closed convex set in $\mathbb{R}^d$ and the origin lies in A°.
2. If $A \subseteq B$ then $B^\circ \subseteq A^\circ$.
3. $\left(\cup_{i \in I} A_i \right)^\circ = \cap_{i \in I} A_i^\circ$.
4. $(\alpha A)^\circ = \alpha^{-1} A^\circ$ for $\alpha > 0$.

Polarity

Let $A \subset \mathbb{R}^d$ be a non-empty set. The set

$$A^\circ = \bigcap_{\mathbf{x} \in A} \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1\}.$$

is called the polar of A.

Properties of the polar set

1. A° is a closed convex set in $\mathbb{R}^d$ and the origin lies in A°.
2. If $A \subseteq B$ then $B^\circ \subseteq A^\circ$.
3. $\left(\cup_{i \in I} A_i \right)^\circ = \cap_{i \in I} A_i^\circ$.
4. $(\alpha A)^\circ = \alpha^{-1} A^\circ$ for $\alpha > 0$.
5. $A \subseteq (A^\circ)^\circ$. (Homework problem.)

Polarity

Let $A \subset \mathbb{R}^d$ be a non-empty set. The set

$$A^\circ = \bigcap_{\mathbf{x} \in A} \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1\}.$$

is called the polar of A.

Properties of the polar set

1. A° is a closed convex set in $\mathbb{R}^d$ and the origin lies in A°.
2. If $A \subseteq B$ then $B^\circ \subseteq A^\circ$.
3. $\left(\cup_{i \in I} A_i\right)^\circ = \cap_{i \in I} A_i^\circ$.
4. $(\alpha A)^\circ = \alpha^{-1} A^\circ$ for $\alpha > 0$.
5. $A \subseteq (A^\circ)^\circ$. (Homework problem.)
6. (Homework problem.) If $A = \operatorname{conv}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is a polytope, then
$$A^\circ = \bigcap_{i \in [n]} \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{v}_i \rangle \leq 1\}.$$

Example and Bipolar Theorem

Example

$(\mathbb{R}^d)^\circ = \{\mathbf{o}\}$, $\{\mathbf{o}\}^\circ = \mathbb{R}^d$ and $(\mathbf{B})^\circ = \mathbf{B}$, where $\mathbf{B} \subset \mathbb{R}^d$ is the unit ball with center at the origin $\mathbf{o}$. (Homework problem.)

Example and Bipolar Theorem

Example

$(\mathbb{R}^d)^\circ = \{\mathbf{o}\}$, $\{\mathbf{o}\}^\circ = \mathbb{R}^d$ and $(\mathbf{B})^\circ = \mathbf{B}$, where $\mathbf{B} \subset \mathbb{R}^d$ is the unit ball with center at the origin $\mathbf{o}$. (Homework problem.)

Bipolar theorem

Let $A \subset \mathbb{R}^d$ be a closed convex set containing the origin. Then $A = (A^\circ)^\circ$.

Proof of bipolar theorem

Bipolar theorem

Let $A \subset \mathbb{R}^d$ be a closed convex set containing the origin. Then $A = (A^\circ)^\circ$.

By 5-th property of polar sets, we have $A \subseteq (A^\circ)^\circ$. Let us show that $(A^\circ)^\circ \subseteq A$. Consider a point $\mathbf{a} \notin A$. Let us show that $\mathbf{a} \notin A$.

Proof of bipolar theorem

Bipolar theorem

Let $A \subset \mathbb{R}^d$ be a closed convex set containing the origin. Then $A = (A^\circ)^\circ$.

By 5-th property of polar sets, we have $A \subseteq (A^\circ)^\circ$. Let us show that $(A^\circ)^\circ \subseteq A$. Consider a point $\mathbf{a} \notin A$. Let us show that $\mathbf{a} \notin A$.
Since A is a closed convex set, by the weak version of the separation theorem, the set A and the one-point set $\{\mathbf{a}\}$ are strictly separated by a hyperplane $h := \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle = \alpha\}$, where $\mathbf{y}$ is a non-zero vector and $\alpha \in \mathbb{R}$.
Since the origin lies in A, we have that $\alpha \neq 0$. Therefore, we can redefine the hyperplane as

$$h := \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle = 1\}.$$

Proof of bipolar theorem

Bipolar theorem

Let $A \subset \mathbb{R}^d$ be a closed convex set containing the origin. Then $A = (A^\circ)^\circ$.

By 5-th property of polar sets, we have $A \subseteq (A^\circ)^\circ$. Let us show that $(A^\circ)^\circ \subseteq A$. Consider a point $\mathbf{a} \notin A$. Let us show that $\mathbf{a} \notin A$.
Since A is a closed convex set, by the weak version of the separation theorem, the set A and the one-point set $\{\mathbf{a}\}$ are strictly separated by a hyperplane $h := \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle = \alpha\}$, where $\mathbf{y}$ is a non-zero vector and $\alpha \in \mathbb{R}$.
Since the origin lies in A, we have that $\alpha \neq 0$. Therefore, we can redefine the hyperplane as

$$h := \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle = 1\}.$$

Again since the origin lies in A, we have that $\langle \mathbf{a}', \mathbf{y} \rangle < 1$ for any $\mathbf{a}' \in A$ and $\langle \mathbf{a}, \mathbf{y} \rangle > 1$.

Proof of bipolar theorem

Bipolar theorem

Let $A \subset \mathbb{R}^d$ be a closed convex set containing the origin. Then $A = (A^\circ)^\circ$.

By 5-th property of polar sets, we have $A \subseteq (A^\circ)^\circ$. Let us show that $(A^\circ)^\circ \subseteq A$. Consider a point $\mathbf{a} \notin A$. Let us show that $\mathbf{a} \notin A$.
Since A is a closed convex set, by the weak version of the separation theorem, the set A and the one-point set $\{\mathbf{a}\}$ are strictly separated by a hyperplane $h := \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle = \alpha\}$, where $\mathbf{y}$ is a non-zero vector and $\alpha \in \mathbb{R}$. Since the origin lies in A, we have that $\alpha \neq 0$. Therefore, we can redefine the hyperplane as

$$h := \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle = 1\}.$$

Again since the origin lies in A, we have that $\langle \mathbf{a}', \mathbf{y} \rangle < 1$ for any $\mathbf{a}' \in A$ and $\langle \mathbf{a}, \mathbf{y} \rangle > 1$.
Thus $\mathbf{y} \in A^\circ$ and $\mathbf{a} \notin (A^\circ)^\circ$, we are done.