Extreme points and polytopes

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Extreme points

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Lemma

Let $A \subset \mathbb{R}^d$ be a non-empty convex set and $f(\mathbf{x}) = \langle \mathbf{c}, \mathbf{x} \rangle$ be a linear functional such that $\mathbf{c}$ is distinct from the origin.

1. If f attains maximum on A at a point $\mathbf{a} \in A$ (that is, $f(\mathbf{a}) > f(\mathbf{b})$ for any $\mathbf{b} \in A \setminus \{\mathbf{a}\}$), then $\mathbf{a}$ is an extreme point of A.
2. (Homework problem.) If f attains maximum equal to α on A and $B = \{\mathbf{x} \in \mathbb{R}^d : f(\mathbf{x}) = \alpha\} \cap A$ is a set with an extreme point $\mathbf{a} \in \operatorname{ex} B$, then $\mathbf{a}$ is an extreme point of A.

Proof of Lemma (part 1)

Suppose that $\mathbf{a} = (\mathbf{b}_1 + \mathbf{b}_2)/2$ for some $\mathbf{b}_1, \mathbf{b}_2 \in A$. Since

$$\langle \mathbf{a}, \mathbf{c} \rangle = \frac{1}{2}\big(\langle \mathbf{b}_1, \mathbf{c} \rangle + \langle \mathbf{b}_2, \mathbf{c} \rangle\big)$$

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we have $\langle \mathbf{a}, \mathbf{c} \rangle = \langle \mathbf{b}_1, \mathbf{c} \rangle = \langle \mathbf{b}_2, \mathbf{c} \rangle$. Therefore, $\mathbf{a} = \mathbf{b}_1 = \mathbf{b}_2$.

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1) If $\mathbf{x} \in \partial K$, then consider a supporting hyperplane at $\mathbf{x}$ and $K' = K \cap h$. Clearly, K' is a compact convex set. Therefore, $x \in \operatorname{conv} \operatorname{ex} K'$.

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2) If $\mathbf{x} \notin \partial K$, then consider any line through $\mathbf{x}$ that intersects the ∂K at points $\mathbf{a}, \mathbf{b}$. By 1), we have $\mathbf{a}, \mathbf{b} \in \operatorname{conv} \operatorname{ex} K$ and thus $\mathbf{x} \in \operatorname{conv} \operatorname{ex} K$.

Polyhedra and polytopes

Let $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{R}^d$ are non-zero vectors and $\beta_1, \ldots, \beta_n \in \mathbb{R}$. The set

$$\bigcap_{i=1}^{n} \left\{ \mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{c}_i \rangle \leq \beta_i \right\}$$

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is called a polyhedron. An extreme point of a polyhedron is called a vertex. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^d$. The set $\operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is called a polytope.

Polyhedron/polytope theorem

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Lemma

Let

$$P = \bigcap_{i=1}^{m} \left\{ \mathbf{x} \in \mathbb{R}^d : \langle \mathbf{c}_i, \mathbf{x} \rangle \le \beta_i \right\}$$

be a polyhedron, where $\mathbf{c}_i \in \mathbb{R}^d \setminus \{\mathbf{o}\}$ and $\beta_i \in \mathbb{R}$ for $i = 1, \dots, m$. For $\mathbf{u} \in P$, let

$$C(\mathbf{u}) = \{\mathbf{c}_i : \langle \mathbf{c}_i, \mathbf{u} \rangle = \beta_i\}.$$

Then $\mathbf{u}$ is a vertex of P iff $C(\mathbf{u})$ linearly spans the vector space $\mathbb{R}^d$.

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where $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is a vectors of variables.

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Since $\mathbf{x} = \mathbf{u}$ and $\mathbf{x} = \mathbf{v}$ are solutions of this system, these points $\mathbf{u}$ and $\mathbf{v}$ must coincide (otherwise, the dimension of the solutions of the system of equations at least 1).

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Therefore, any vertex $\mathbf{u}$ of P has a unique set $C(\mathbf{u}) \subset \{\mathbf{c}_1, \ldots, \mathbf{c}_m\} = C$. Since the set C has finitely many subsets, there are finitely many vertices. We are done.

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It is enough to prove that for $|\varepsilon| \leq \delta(P)$, where $\delta(P) > 0$ is sufficiently small, the point $\mathbf{u}(\varepsilon) = \mathbf{u} + \varepsilon\mathbf{w} \in P$, that is

$$\langle \mathbf{c}_i, \mathbf{u}(\varepsilon)\rangle = \langle \mathbf{c}_i, \mathbf{u}\rangle + \varepsilon\langle \mathbf{c}_i, \mathbf{w}\rangle \leq \beta_i \quad (*)$$

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Choosing $\delta(P) = \min \delta_i$, we have that (*) holds for all $\mathbf{c}_i \in C$.

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Choosing $\delta(P) = \min \delta_i$, we have that (*) holds for all $\mathbf{c}_i \in C$. Therefore, $\mathbf{u}(\varepsilon) \in P$ for $|\varepsilon| \le \delta(P)$, and hence, $\mathbf{u}$ is not a vertex.

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$$\Big\{ \langle \mathbf{c}_i, \mathbf{x} \rangle = \beta_i, \quad \text{for } \mathbf{c}_i \in C(\mathbf{u}),$$

where $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is a vector of variables.

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where $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is a vector of variables. Since $C(\mathbf{u})$ spans $\mathbf{R}^d$, the dimension of the solutions is at most 0. But $\mathbf{u}, \mathbf{a}, \mathbf{b}$ are solutions of this system.

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Now suppose that for a point $\mathbf{u} \in \mathbb{R}^d$, we have that $C(\mathbf{u})$ linearly spans $\mathbb{R}^d$. Let us show that $\mathbf{u}$ is a vertex, that is, an extreme point of P.

Assume that $\mathbf{u} = (\mathbf{a} + \mathbf{b})/2$ for some $\mathbf{a}, \mathbf{b} \in P$. Since for $\mathbf{c}_i \in C(u)$, we have

$$\beta_i = \langle \mathbf{c}_i, \mathbf{u} \rangle = \frac{1}{2}\langle \mathbf{c}_i, \mathbf{a} \rangle + \frac{1}{2}\langle \mathbf{c}_i, \mathbf{b} \rangle \leq \frac{1}{2}\beta_i + \frac{1}{2}\beta_i = \beta_i.$$

Hence $\langle \mathbf{c}_i, \mathbf{a} \rangle = \langle \mathbf{c}_i, \mathbf{b} \rangle = \langle \mathbf{c}_i, \mathbf{u} \rangle = \beta_i$ for all $\mathbf{c}_i \in C(\mathbf{u})$.

Consider the system of linear equations

$$\left\{ \langle \mathbf{c}_i, \mathbf{x} \rangle = \beta_i, \quad \text{for } \mathbf{c}_i \in C(\mathbf{u}), \right.$$

where $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is a vector of variables. Since $C(\mathbf{u})$ spans $\mathbf{R}^d$, the dimension of the solutions is at most 0. But $\mathbf{u}, \mathbf{a}, \mathbf{b}$ are solutions of this system. Hence $\mathbf{u} = \mathbf{a} = \mathbf{b}$.

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Hence K° is a bounded polyhedron, and therefore it is a polytope (as we proved earlier), that is, $K^\circ = \text{conv}\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$. Using 6th property of polar sets and the Bipolar theorem, we finally obtain that

$$K = (K^\circ)^\circ = \cap_{j=1}^{m} \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{w}_j, \mathbf{x} \rangle \leq 1\}$$

is a bounded polyhedron.