Circle covering theorem

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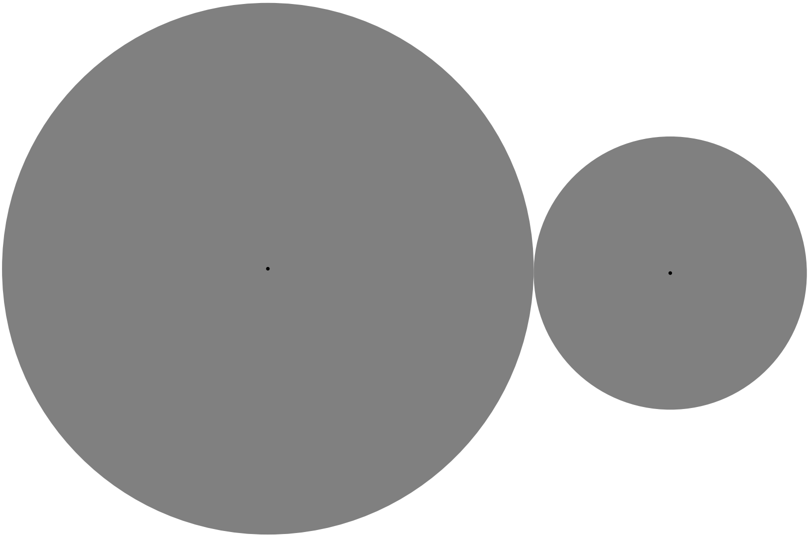
Theorem (Goodman & Goodman, 1945)

A finite non-separable collection $\mathcal{K}$ of discs can be covered by a disk of radius equal to the total radius of the disks of $\mathcal{K}$.

Examples



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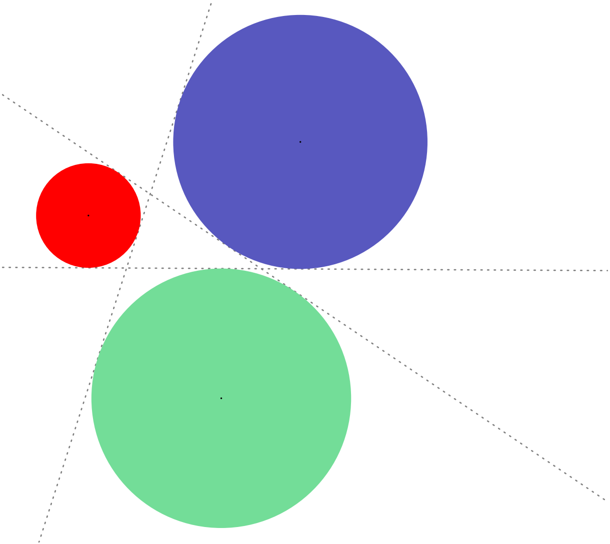

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1-dimensional version

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We can apply a more conceptual idea.

- Find two intersecting segments $B_{r_1}(a_1)$ and $B_{r_2}(a_2)$ and find the center of mass of a_1 and a_2 with weights r_1 and r_2, that is, the point $a = \frac{r_1 a_1 + r_2 a_2}{r_1 + r_2}$.

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- Verify that $B_{r_1+r_2}(a)$ covers $B_{r_1}(a_1)$ and $B_{r_2}(a_2)$ and induct.

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- Verify that $B_{r_1+r_2}(a)$ covers $B_{r_1}(a_1)$ and $B_{r_2}(a_2)$ and induct.
- Finally we obtain that the segment $B_{r_1+\dots+r_n}\left(\frac{r_1\mathbf{a}_1+\dots+r_n\mathbf{a}_n}{r_1+\dots+r_n}\right)$ is desired.

Proof of the Goodman^2 Theorem

- Find the center **a** of mass of centers of the discs $B_{r_1}(\mathbf{a}_1), \ldots, B_{r_n}(\mathbf{a}_n)$ with weights $r_1, \ldots, r_n$, respectively. Let us verify that the ball $B_{r_1+\cdots+r_n}\left(\frac{r_1\mathbf{a}_1+\cdots+r_n\mathbf{a}_n}{r_1+\cdots+r_n}\right)$ satisfies the conclusion of the theorem of Goodman and Goodman.

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- Consider any line ℓ through $\mathbf{a}$ and project balls on that line. The projections are segments of lengths $2r_1, \ldots, 2r_n$ and with centers $\mathbf{a}'_1, \ldots, \mathbf{a}'_n$. Clearly, the collection of obtained segments is non-separable

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- The center of mass of new segments coincides with $\mathbf{a}$. Thus the segment with center $\mathbf{a}$ and radius $r_1 + \cdots + r_n$ coveres the collection of segments.
- Since this holds for any line through $\mathbf{a}$, the ball $B_{r_1+\cdots+r_n}(\mathbf{a})$ covers the initial collection of balls.