Bang's covering theorem

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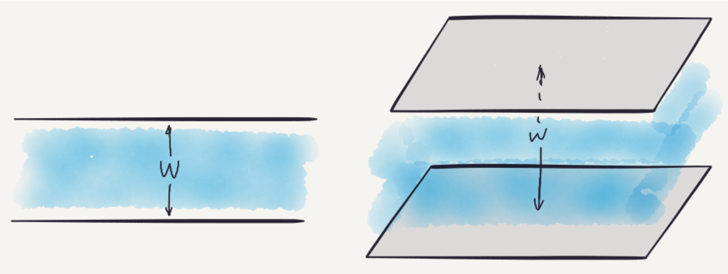
Plank



A *plank* (or *slab*, or *strip*) of width w is a part of $\mathbb{R}^3$ that lies between two parallel hyperplanes at distance w.

Tarski's plank problem

The *width* of a convex body K is the smallest width of a plank covering K.

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Theorem (Moese, 1933; Tarski, 1933)

If a disc is covered by planks, then the total width of the planks is at least the ~~diameter~~ width of the disc.

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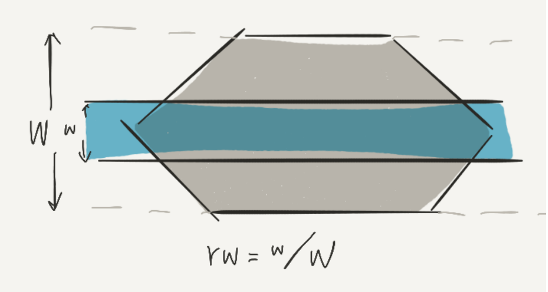
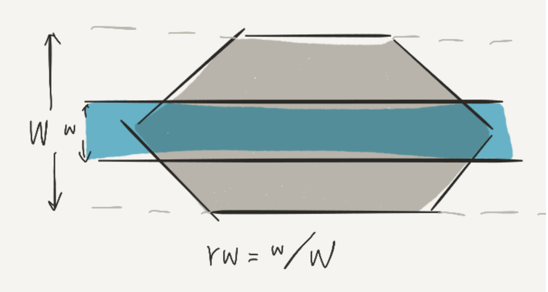
Theorem (Bang, 1950, 1951)

The previous theorem is true for **any convex body**.

Bang's plank problem



Bang's plank problem



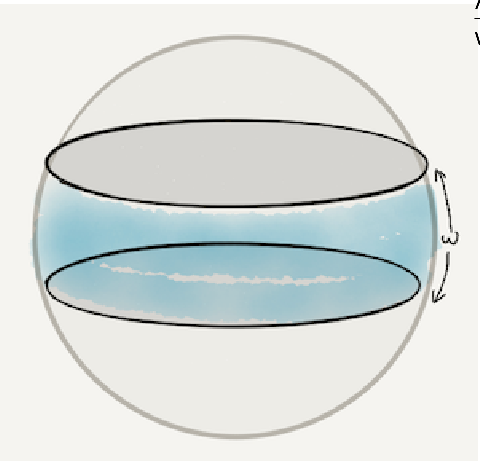
Conjecture (Bang, 1951)

If a convex body K is covered by planks, then the total **relative** width is at least 1.

Theorem (Ball, 1991)

Conjecture is true for **any symmetric convex body**.

Zone



Assumption: A sphere S is unit and with center in the origin of $\mathbb{R}^3$.

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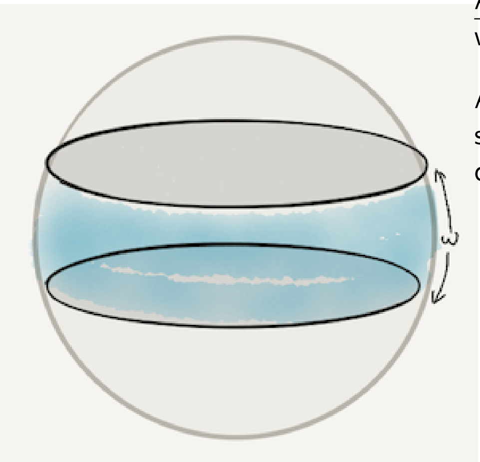

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A *zone* of width ω is a part of a sphere that lies within spherical distance $\omega/2$ of a given great circle.

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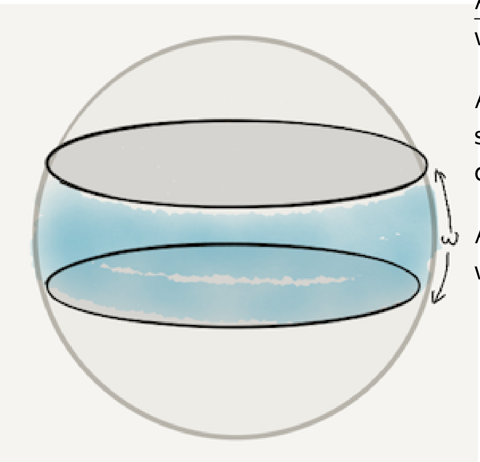

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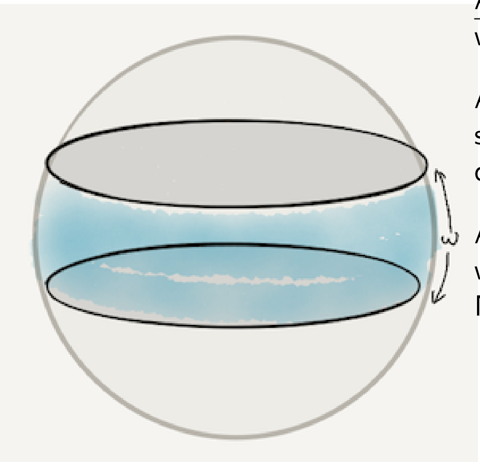

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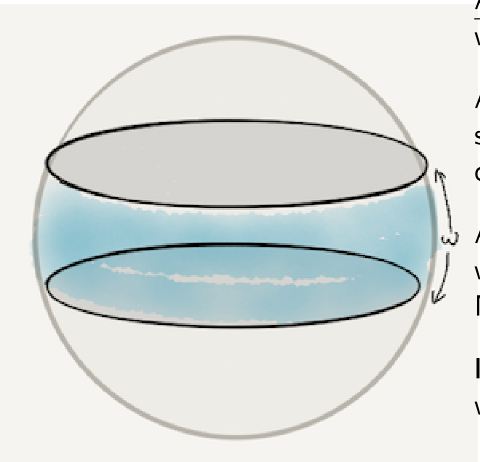

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A zone is the intersection of a sphere with an o-symmetric plank P. Notation for the zone: $Z(P)$.

If the width of $Z(P)$ is 2α, then the width of P is $2\sin\alpha$.

Fejes Tóth's zone conjecture

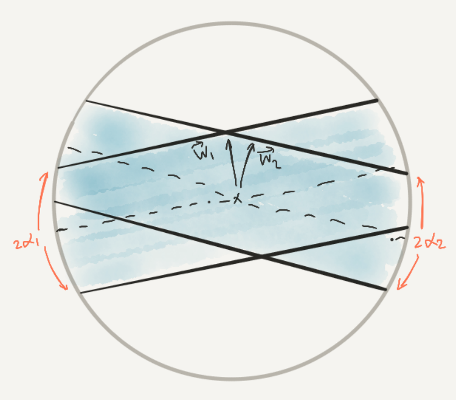
Conjecture (Fejes Tóth, 1973)

The total width of any set of zones covering the sphere is at least π.

Theorem (Jiang & AP, 2017; Ortega–Moreno, 2021: the same width case)

László Fejes Tóth's zone conjecture is true.

Proof of Fejes Tóth's zone conjecture



Assumption: Zones $Z_1 = Z(P_1), \dots, Z_n = Z(P_n)$ have widths $2\alpha_1, \dots, 2\alpha_n$ and $\alpha_1 + \dots + \alpha_n < \pi/2$.

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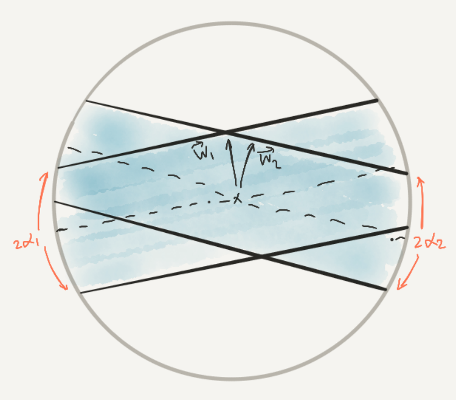

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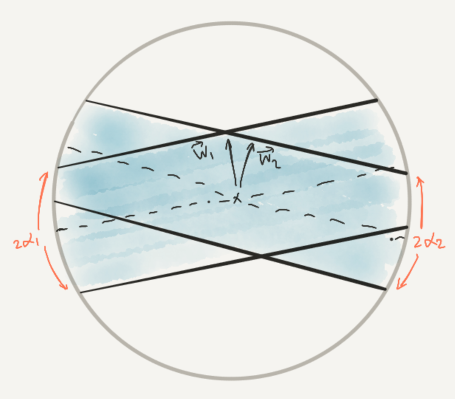

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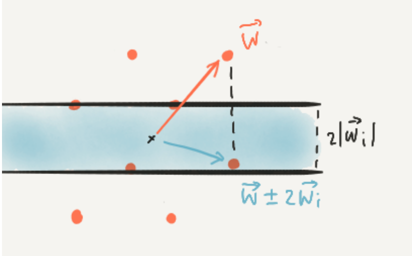
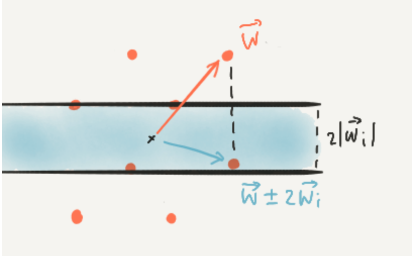
Denote by $\mathbf{w}_i$ a vector orthogonal to the boundary hyperplanes of P_i of length $\sin \alpha_i$

Bang's discrete set



Consider the set

$$L = \left\{ \sum_{i=1}^{n} \pm \mathbf{w}_i \right\}.$$

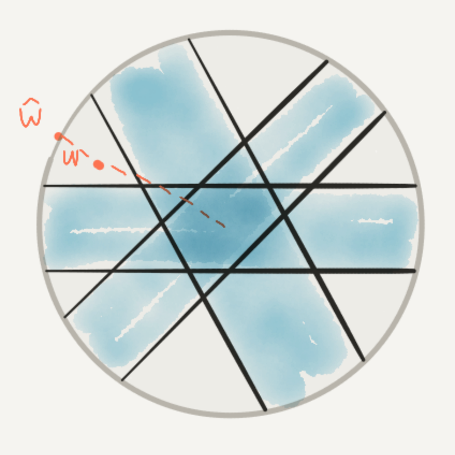


Consider the set

$$L = \left\{ \sum_{i=1}^{n} \pm \mathbf{w}_i \right\}.$$

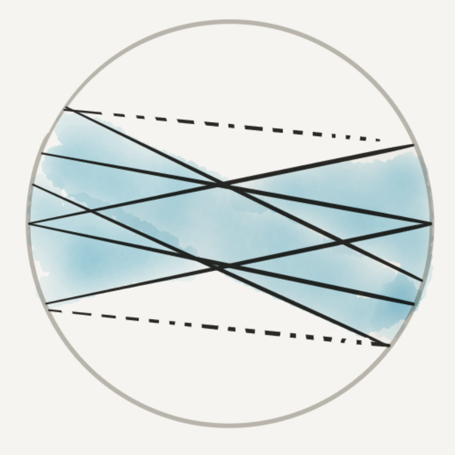
Let $\mathbf{w} = \mathbf{w}_1 + \cdots + \mathbf{w}_n$ has the maximal norm among points of L. Then it does not lie in the interior of P_i!

Projection



If $|\mathbf{w}| < 1$, then $\mathbf{w}/|\mathbf{w}|$ is not covered by ~~planks~~ zones.

Merging



If $\mathbf{w} \geq 1 = \sin(\pi/2) > \sin(\sum_{i=1}^{n} \alpha_i)$, then let us tackle to merge zones.

Key Lemma

Lemma (Jiang & AP, 2017)

Let $I \subseteq [n]$ is a non-empty subset of indexes such that

$$\left|\sum_{i\in I} \mathbf{w}_i\right| \geq \sin\left(\sum_{i\in I} \alpha_i\right) \qquad (1)$$

$$\left|\sum_{i\in I\setminus\{j\}} \mathbf{w}_i\right| \leq \sin\left(\sum_{i\in I\setminus\{j\}} \alpha_i\right) \text{ for any } j \in I,$$

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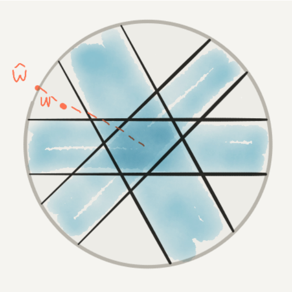
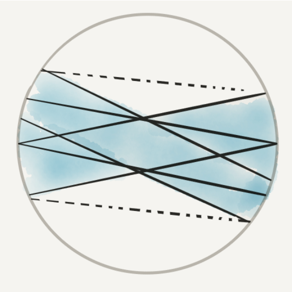
Find a minimal subset $I \subseteq [n]$ satisfying '**stricted**' (1).
Existance: $[n]$ satisfies '**stricted**' (1).
Cardinality: $|I| > 1$ because $|\mathbf{w}| = \sin \alpha_i$.

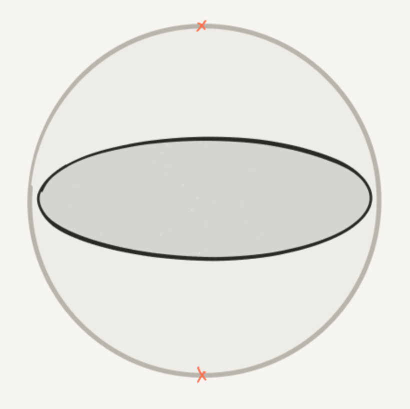


If $|\mathbf{w}| < 1$, then $\mathbf{w}/|\mathbf{w}|$ is not covered.

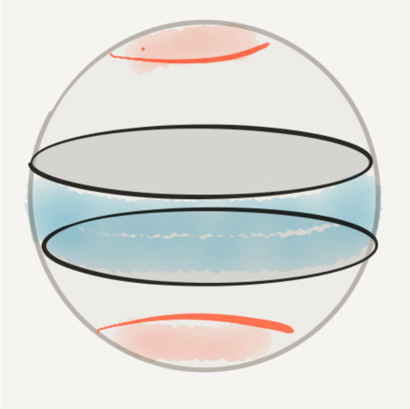


If $|\mathbf{w}| \geq 1$, then we can merge a few zones.

Projective Duality



Pair of antipodal points
Great sphere



Pair of antipodal caps of radii α
Zone of width 2α

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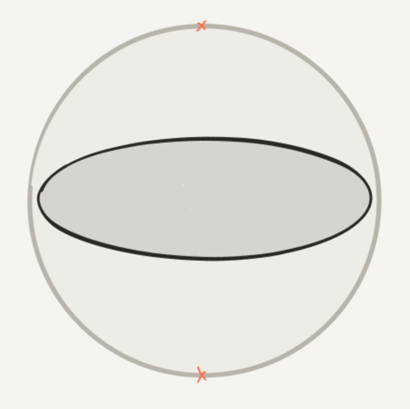

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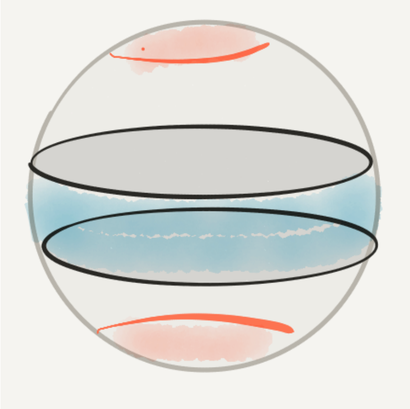

Pair of antipodal caps of radii α
Zone of width 2α

A pair of antipodal points $\mathbf{x}$ avoids a zone iff its dual pair of caps avoids the great dual to $\pm\mathbf{x}$.

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Dual to FT's zone conjecture

If the total spherical radius of caps is at most $\pi/2$, then there exists a great circle non-intersecting the caps.