The Brunn–Minkowski inequality

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Let J be a bounded subset of $\mathbb{R}$. If $\lambda^*(J) = \lambda_*(J)$, then we call J measurable. We call $\lambda(J) = \lambda^*(J) = \lambda_*(J)$ the length of J.

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$$\lambda_*(J) = \lambda^*(B) - \lambda^*(B \setminus J).$$

If $\lambda^*(J) = \lambda_*(J)$, then J is called measurable.

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Suppose that the sides of A are equal to $\alpha_1, \dots, \alpha_d$ and the sides of B are equal to $\beta_1, \dots, \beta_d$. Then the sides of $A+B$ are equal to $\alpha_1+\beta_1, \dots, \alpha_d+\beta_d$.

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This inequality is your homework.

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- The base of induction $(n_a + n_b = 2)$ is proved.
- Now, suppose that $n_a + n_b = k > 2$ and for smaller number of boxes the inequality is proved. Without loss of generality $n_a \geq 2$.
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- Consider a hyperplane H separating some two boxes in A and choose properly a translate of B. W.l.o.g. H passes through the origin. W.l.o.g. the translation vector is $\mathbf{o}$.

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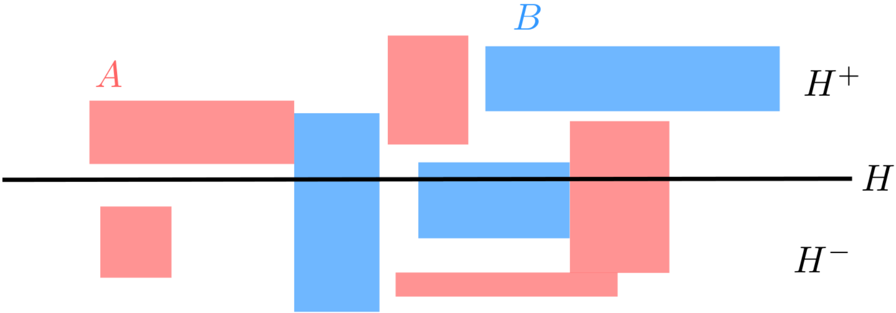

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- Let $A_\pm = H^\pm \cap A$ and $B_\pm = H^\pm \cap B$.
- Suppose that
$$\frac{\operatorname{vol} A_+}{\operatorname{vol} A_-} = \frac{\operatorname{vol} B_+}{\operatorname{vol} B_-} = \frac{\lambda}{1-\lambda}.$$
- Since the total number of boxes in H^+ is less than $k = n_a + n_b$, we may apply the induction hypothesis for A_+ and B_+, that is,
$$\operatorname{vol}^{1/d}(A_+ + B_+) \geq \operatorname{vol}^{1/d} A_+ + \operatorname{vol}^{1/d} B_+ = \lambda^{1/d}(\operatorname{vol}^{1/d} A + \operatorname{vol}^{1/d} B).$$
Hence
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- Remark that $A_\pm + B_\pm \subseteq H_\pm$. Therefore,
$$\operatorname{vol}(A+B) \geq \operatorname{vol}(A_+ + B_+) + \operatorname{vol}(A_+ + B_+) \geq (\operatorname{vol}^{1/d} A + \operatorname{vol}^{1/d} B)^d.$$

Brunn–Minkowskii in general case

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- Therefore,

$$|\mathcal{A} + \mathcal{B}| \geq |\mathcal{A}| + |\mathcal{B}|.$$

  It implies that $|A + B| \geq |A| + |B|$.