The Dancer–Grunbaum theorem

Alexander Polyanskii
http://polyanskii.com/

Moscow Institute of Physics and Technology

Antipodal sets

A finite set A of points in $\mathbb{R}^d$ is called antipodal if for any two distinct points $a_1, a_2 \in A$ there are two distinct hyperplanes H_1 and H_2 such that $a_1 \in H_1$, $a_2 \in H_2$ and $A \subset \text{conv}(H_1 \cup H_2)$.

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Dancer–Grunbaum theorem

The size of an antipodal set in $\mathbb{R}^d$ does not exceed 2^d.

Proof of the DG theorem

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- For each $a \in A$, consider a homothetic copy B_a of B with center at a and coefficient $1/2$.
- Since A is an antipodal set, B_{a_1} and B_{a_2} are separated.
- Moreover, $B_a \subset B$.
- Therefore,

$$\operatorname{vol} B \geq \sum_{a \in A} \operatorname{vol}(B_a) = |A| \cdot \operatorname{vol}\left(B/2\right) = |A| \frac{\operatorname{vol}(B)}{2^d}.$$

The last inequality finishes the proof.

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Theorem

The size of an equidistant set in a d-dimensional Banach space does not exceed 2^d.

Proof

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- By the triangle inequality, for any $x, y \in C$, the norm of $x - y$ is at most 2, that is , $C - C \subseteq 2B$.
- By the Brunn-Minkowski Inequality, we have

$$\mathrm{vol}^{1/d}(2B) \geq \mathrm{vol}^{1/d}(C - C) \geq \mathrm{vol}^{1/d}(C) + \mathrm{vol}^{1/d}(-C)$$

$$= 2\,\mathrm{vol}^{1/d}\, C = 2(|A|\,\mathrm{vol}(B/2))^{1/d}.$$

The last inequality implies that $|A| \leq 2^d$.