Planar graphs

Alexander Polyanskii
http://polyanskii.com/

Moscow Institute of Physics and Technology

Definition

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Plane graph

A plane graph := a planar graph + its drawing.

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Notation

V is the set of vertices, E is the set of edges, F is the set of faces.

Planar graphs on spheres

A graph can be drawn on a plane **iff** it can be drawn on a sphere.

Proof

Consider a spherical projection!

Euler's Formula

If a plane graph has k components, v vertices, e edges, and f faces, then

$$v - e + f - k = 1.$$

In particular, if the graph is connected (that is, $k = 1$), then

$$v - e + f = 2.$$

Proof

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- Delete any edge of the drawing of the planar graph. There are two options:
    1. The edge separates two different faces. Then $f \to f - 1$, $k \to k$.
    2. The edge is incident to the same face. Then $f \to f$ and $k \to k + 1$.

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Observation

Euler's Formula holds even if a plane graph has a multiple edges or loops!

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Corollary

K_5 and $K_{3,3}$ are not planar graphs.

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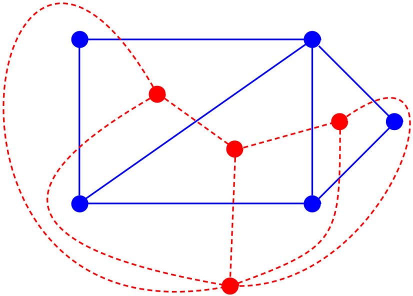
The second part of this lemma is a homework problem.

Dual graph



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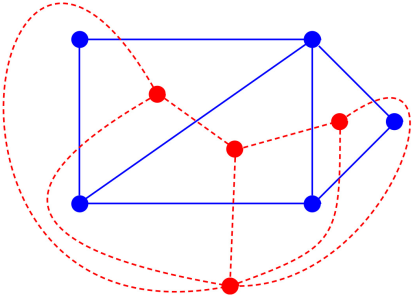

One of the most fundamental properties of plane graphs is the existence of a dual graph. A dual G^* of a plane graph G is a plane graph having a vertex in each face of G. Every edge e of G has a corresponding dual edge e^* in G^*: If F and F' are the faces on the two sides of e then e^* connects the vertices of G^* in F and F'.

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Lemma

If that $v \geq 3$, then $\mathrm{cr}(G) \geq e - 3v + 6$.

It is another homework problem.

What is the order of the lower bound in the previous lemma? It is roughly e (that could be about v^2).

Crossing Lemma

If G is a graph with e edges and v such that $e \geq 4v$, then

$$\mathrm{cr}(G) \geq \frac{1}{64} \cdot \frac{e^3}{v^2}$$

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This lemma implies that if $e = c\dot{v}^2$ for some constant $c > 0$, then $\text{cr}(G) \geq Cv^4$ for some $C > 0$. This bound is much better than the trivial bound v^2.

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Since

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we have

$$p^4 \,\mathrm{cr}(G) \ge p^2 e - 3pv + 6 \Rightarrow \mathrm{cr}(G) \ge \frac{e}{p^2} - \frac{3v}{p^3}.$$

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Choosing $p = 4v/e \leq 1$, we get

$$\mathrm{cr}(G) \geq \frac{e^3}{16v^2} - \frac{3e^3}{64v^2} = \frac{e^3}{64v^2}$$