Planar graphs - II

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Sylvester–Gallai theorem

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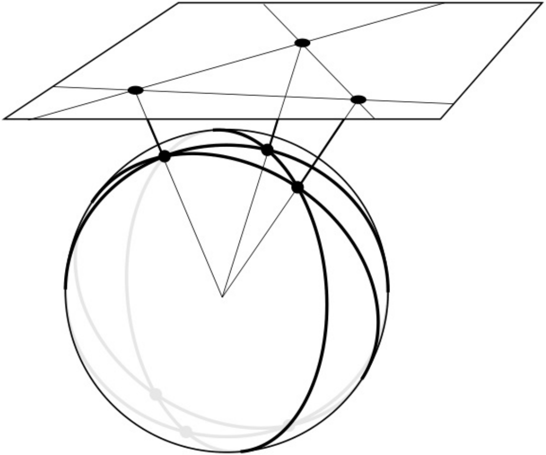
Proof. First, let us project all lines onto the sphere S using the central projection. The images of the lines are great circles of the sphere.

Projection



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- Hence the size of every face at least 3, and so $e \leq 3v - 6$.
- There is a vertex of degree less than 6! It must belong to exactly two circles and the corresponding plane point belongs to exactly two lines!