Discharging method

Alexander Polyanskii
http://polyanskii.com/

Moscow Institute of Physics and Technology

Discharging method: Strategy

For a connected planar graph Γ, consider Euler's Formula

$$|V(\Gamma)| - |E(\Gamma)| + |F(\Gamma)| = 2.$$

Multiplying the equality by (-2) and using $2|E(\Gamma)| = \sum_{v \in V} \deg v = \sum_{f \in F} \deg f$, we can rewrite Euler's Formula in the following way

Discharging method: Strategy

For a connected planar graph Γ, consider Euler's Formula

$$|V(\Gamma)| - |E(\Gamma)| + |F(\Gamma)| = 2.$$

Multiplying the equality by (-2) and using $2|E(\Gamma)| = \sum_{v \in V} \deg v = \sum_{f \in F} \deg f$, we can rewrite Euler's Formula in the following way

$$\sum_{v \in V(\Gamma)} (\alpha \deg v - 2) + \sum_{f \in F(\Gamma)} ((1-\alpha) \deg f - 2) = -4. \quad (*)$$

Discharging method: Strategy

$$\sum_{v \in V(\Gamma)} (\alpha \deg v - 2) + \sum_{f \in F(\Gamma)} ((1-\alpha)\deg f - 2) = -4. \quad (*)$$

- Choosing different values of α, we obtain different useful equalities. Let us assign charges to the vertices and the faces of Γ according to the left-hand side of (*).

Discharging method: Strategy

$$\sum_{v \in V(\Gamma)} (\alpha \deg v - 2) + \sum_{f \in F(\Gamma)} ((1-\alpha) \deg f - 2) = -4. \quad (*)$$

- Choosing different values of α, we obtain different useful equalities. Let us assign charges to the vertices and the faces of Γ according to the left-hand side of (*).
- Of course, the total charge is negative. Then we suppose the contrary to the desired statement of the problem. Using that and the properties of the graph given in a problem, we rearrange the charges in a way that the final charge of every element (a vertex or a face) becomes non-negative. Usually, this contradiction finishes the proof.

Example

Claim

Let G be a connected plane graph with the minimum degree at least 5 such that all its faces are triangles, and there are no two adjacent vertices of degree 5. Then there is a face with degrees of vertices 5, 6 and 6, respectively.

Example

Claim

Let G be a connected plane graph with the minimum degree at least 5 such that all its faces are triangles, and there are no two adjacent vertices of degree 5. Then there is a face with degrees of vertices 5, 6 and 6, respectively.

Proof. Choosing $\alpha = 1/3$ and multiplying (*) by 3, we have

$$\sum_{v \in V(\Gamma)} (\deg v - 6) + \sum_{f \in F(\Gamma)} (2 \deg f - 6) = -12.$$

Since every face is a triangle, we have

$$\sum_{v \in V(\Gamma)} (\deg v - 6) = -12.$$

Let us give charge $\deg v - 6$ to a vertex v.

Example

Suppose that there is no triangle with degrees (5,6,6). Hence every vertex of degree 5 is incident to at least 3 vertices of degree at least 7.

Example

Suppose that there is no triangle with degrees (5,6,6). Hence every vertex of degree 5 is incident to at least 3 vertices of degree at least 7.

Since there are no edges with degrees (5,5), every vertex of degree n is incident to at most $n/2$ vertices of degree 5. In particular every vertex of degree 7 is incident to at most 3 vertices of degree 5.

Example

Suppose that there is no triangle with degrees (5,6,6). Hence every vertex of degree 5 is incident to at least 3 vertices of degree at least 7.

Since there are no edges with degrees (5,5), every vertex of degree n is incident to at most $n/2$ vertices of degree 5. In particular every vertex of degree 7 is incident to at most 3 vertices of degree 5.

Discharging rules:

- Every vertex of degree at least 7 give charge $1/3$ to its neighboring vertex of degree 5.

Example

Suppose that there is no triangle with degrees (5,6,6). Hence every vertex of degree 5 is incident to at least 3 vertices of degree at least 7.

Since there are no edges with degrees (5,5), every vertex of degree n is incident to at most $n/2$ vertices of degree 5. In particular every vertex of degree 7 is incident to at most 3 vertices of degree 5.

Discharging rules:

- Every vertex of degree at least 7 give charge $1/3$ to its neighboring vertex of degree 5.

Then a vertex of degree 5 will have charge at least 0, a vertex of degree 6 will have charge 0, a vertex of degree 7 will have charge at least 0, a vertex of degree $n \geq 8$ will have charge at least $n - 6 - \frac{n}{6} = \frac{5n-36}{6} \geq \frac{4}{6} > 0$.

Discharging method

We say that a graph Γ is drawn in the plane if the vertices $V(\Gamma)$ are represented by distinct points and the edges $E(\Gamma)$ are represented by (Jordan) arcs, each connecting two vertices and containing no other vertex.

Discharging method

We say that a graph Γ is drawn in the plane if the vertices $V(\Gamma)$ are represented by distinct points and the edges $E(\Gamma)$ are represented by (Jordan) arcs, each connecting two vertices and containing no other vertex.

Quasi-planar graphs

A graph is called quasi-planar if it can be drawn in the plane in such a way that each two edges intersect in at most one inner point and no three edges are pairwise crossing in their inner points.

Main theorem

The number of edges of a quasi-planar graph G with n vertices does not exceed $10n - 20$.

Useful definitions

- Consider the following plane graph Γ'. Its vertices are vertices of Γ and the points of intersection of edges of Γ. Edges of Γ' are pairs of vertices that are end points of parts of curves (=edges) of Γ.

Useful definitions

- Consider the following plane graph Γ'. Its vertices are vertices of Γ and the points of intersection of edges of Γ. Edges of Γ' are pairs of vertices that are end points of parts of curves (=edges) of Γ.
- Denote by V, E, F, V', E', F' the number of vertices, edges and faces of graphs Γ and Γ', respectively. Vertices of $V(\Gamma)$ are called old and the rest are called new.

Useful definitions

- Consider the following plane graph Γ'. Its vertices are vertices of Γ and the points of intersection of edges of Γ. Edges of Γ' are pairs of vertices that are end points of parts of curves (=edges) of Γ.
- Denote by V, E, F, V', E', F' the number of vertices, edges and faces of graphs Γ and Γ', respectively. Vertices of $V(\Gamma)$ are called old and the rest are called new.
- A degree of a vertex is its degree in the new graph Γ'. Clearly, the degree of a new vertex is 4.

Important observation

It is easy to see that $E' - E = 2(V' - V)$, that is,

$$2E' = 2E + 4(V' - V).$$

Important observation

It is easy to see that $E' - E = 2(V' - V)$, that is,

$$2E' = 2E + 4(V' - V).$$

Here is Euler's formula for Γ' multiplied by (-2):

$$-2V' + 2E' - 2F' = -4.$$

Important observation

It is easy to see that $E' - E = 2(V' - V)$, that is,

$$2E' = 2E + 4(V' - V).$$

Here is Euler's formula for Γ' multiplied by (-2):

$$-2V' + 2E' - 2F' = -4.$$

Note that

$$2E' = \sum_{v \in V(\Gamma')} \deg v = 2E' = \sum_{f \in F(\Gamma')} \deg f.$$

Important observation

It is easy to see that $E' - E = 2(V' - V)$, that is,

$$2E' = 2E + 4(V' - V).$$

Here is Euler's formula for Γ' multiplied by (-2):

$$-2V' + 2E' - 2F' = -4.$$

Note that

$$2E' = \sum_{v \in V(\Gamma')} \deg v = 2E' = \sum_{f \in F(\Gamma')} \deg f.$$

Therefore,

Important observation

$$\sum_{v \in V(\Gamma')} (-2) + 2E' + \sum_{f \in F(\Gamma')} (-2) = -4,$$

Important observation

$$\sum_{v\in V(\Gamma')}(-2)+2E'+\sum_{f\in F(\Gamma')}(-2)=-4,$$

$$2E'=\sum_{v\in V(\Gamma')}\deg v=\sum_{f\in F(\Gamma')}\deg f=2E+\sum_{v\in V(\Gamma')\setminus V(\Gamma)}4.$$

Therefore,

Important observation

$$\sum_{v\in V(\Gamma')}(-2)+2E'+\sum_{f\in F(\Gamma')}(-2)=-4,$$

$$2E'=\sum_{v\in V(\Gamma')}\deg v=\sum_{f\in F(\Gamma')}\deg f=2E+\sum_{v\in V(\Gamma')\setminus V(\Gamma)}4.$$

Therefore,

$$2\gamma E+\sum_{v\in V(\Gamma')\setminus V(\Gamma)}(\alpha\deg v-2+4\gamma)+$$

$$\sum_{v\in V(\Gamma)}(\alpha\deg v-2)+\sum_{f\in F(\Gamma')}(\beta\deg f-2)=-4,$$

where $\alpha+\beta+\gamma=1$.

Important observation

$$2\gamma E + \sum_{v \in V(\Gamma') \setminus V(\Gamma)} (\alpha \deg v - 2 + 4\gamma) +$$

$$\sum_{v \in V(\Gamma)} (\alpha \deg v - 2) + \sum_{f \in F(\Gamma')} (\beta \deg f - 2) = -4,$$

where $\alpha + \beta + \gamma = 1$.

Important observation

$$2\gamma E + \sum_{v \in V(\Gamma') \setminus V(\Gamma)} (\alpha \deg v - 2 + 4\gamma) +$$

$$\sum_{v \in V(\Gamma)} (\alpha \deg v - 2) + \sum_{f \in F(\Gamma')} (\beta \deg f - 2) = -4,$$

where $\alpha + \beta + \gamma = 1$. For $v \in V(\Gamma') \setminus V(\Gamma)$, we have $\deg v =$

Important observation

$$2\gamma E + \sum_{v \in V(\Gamma') \setminus V(\Gamma)} (\alpha \deg v - 2 + 4\gamma) +$$

$$\sum_{v \in V(\Gamma)} (\alpha \deg v - 2) + \sum_{f \in F(\Gamma')} (\beta \deg f - 2) = -4,$$

where $\alpha + \beta + \gamma = 1$. For $v \in V(\Gamma') \setminus V(\Gamma)$, we have $\deg v = 4$. We assume that $4\alpha - 2 + 4\gamma = 0$, that is, $\beta = 0.5$.

$$2\gamma E + \sum_{v \in V(\Gamma)} (\alpha \deg v - 2) + \sum_{f \in F(\Gamma')} (0.5 \deg f - 2) = -4,$$

where $\alpha + \gamma = 0.5$.

Choice of the parameters

Choose $\gamma = 0.1$, that is, $\alpha = 0.4$.

$$0.2E + \sum_{v \in V(\Gamma)} (0.4 \deg v - 2) + \sum_{f \in F(\Gamma')} (0.5 \deg f - 2) = -4$$

Choice of the parameters

Choose $\gamma = 0.1$, that is, $\alpha = 0.4$.

$$0.2E + \sum_{v \in V(\Gamma)} (0.4 \deg v - 2) + \sum_{f \in F(\Gamma')} (0.5 \deg f - 2) = -4$$

$$E = -\sum_{v \in V(\Gamma)} (2 \deg v - 10) - \sum_{f \in F(\Gamma')} (2.5 \deg f - 10) - 20.$$

Goal

$$-E = \sum_{v \in V(\Gamma)} (2 \deg v - 10) + \sum_{f \in F(\Gamma')} (2.5 \deg f - 10) + 20.$$

Goal

$$-E = \sum_{v \in V(\Gamma)} (2 \deg v - 10) + \sum_{f \in F(\Gamma')} (2.5 \deg f - 10) + 20.$$

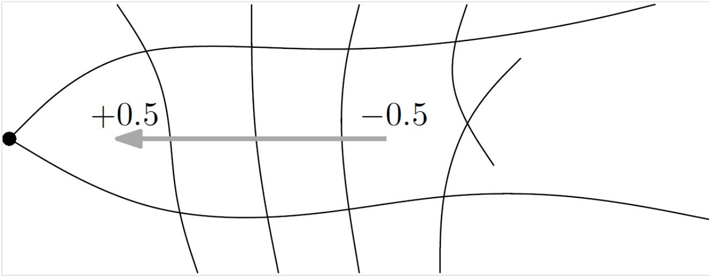
Since we are going to prove $E \leqslant 10V - 20$, we would like to show that

$$-E + 10V - 20 = \sum_{v \in V(\Gamma)} 2 \deg v + \sum_{f \in F(\Gamma')} (2.5 \deg f - 10) \geq 0$$

Charging and discharging

Charging: Let us give to every old vertex v charge $2 \deg v$ and to every face f charge $(2.5 \deg f - 10)$.
Discharging: Let every vertex give charge 2 to incident faces. Then the charge of every vertex becomes 0.

Discharging



Discharging

Let every face $f \in F(\Gamma')$ with $\deg f \geqslant 5$ give charge 0.5 to faces incident to it, that is, to faces that share an edge with f.

Discharging

Let every face $f \in F(\Gamma')$ with $\deg f \geqslant 5$ give charge 0.5 to faces incident to it, that is, to faces that share an edge with f. If the next face is a quadrangle without old vertices (let us call such a face **new** and all the rest **old**), then it gives charge 0.5 to the next face through the opposite face and etc. till this charge reach an old face.

Computation of the total charge

- Charge of every vertex is 0.

Computation of the total charge

- Charge of every vertex is 0.
- Charge of every face of size ≥ 5 is $2.5 \deg f - 10 - 0.5 \deg f = 2 \deg f - 10 \geq 0$.

Computation of the total charge

- Charge of every vertex is 0.
- Charge of every face of size ≥ 5 is $2.5 \deg f - 10 - 0.5 \deg f = 2 \deg f - 10 \geq 0$.
- Charge of every face of size 4 is 0.

Computation of the total charge

- Charge of every vertex is 0.
- Charge of every face of size ≥ 5 is $2.5 \deg f - 10 - 0.5 \deg f = 2 \deg f - 10 \geq 0$.
- Charge of every face of size 4 is 0.
- Initial charge of every triangular face is -2.5. There are no 3-faces with new vertices because of the definition of quasi-planar graphs.
- If 3-face is incident to at least two old vertices, then the final charge of it is > 0.
- If it is incident to one vertex, then its final charge is non-negative. (See the drawing.)