Cauchy's theorem

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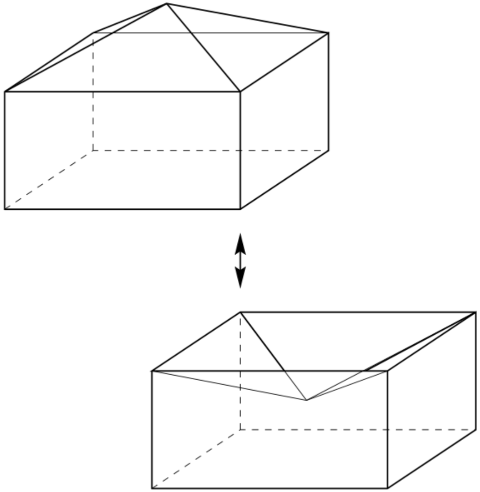
Theorem

If two 3-dimensional convex polyhedra P and Q are combinatorially equivalent with corresponding facets being congruent, then also the angles between corresponding pairs of adjacent facets are equal (and thus P is congruent to Q).

Convexity is important



Proof

- W.l.o.g, consider a plane graph $G = (V, E)$ of boundaries of P and Q. This graph is a simple graph with at least 4 vertices, degree of every vertex is at least 3, and the size of every face is at least 3.
- For every edge $e \in E$, consider corresponding dihedral angles $\alpha_{e,P}$ and $\alpha_{e,Q}$.

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- For every edge $e \in E$, consider corresponding dihedral angles $\alpha_{e,P}$ and $\alpha_{e,Q}$.
- Let us color edges of G according to the following rule:
    - If $\alpha_{e,P} > \alpha_{e,Q}$, then e is colored in blue.
    - If $\alpha_{e,P} < \alpha_{e,Q}$, then e is colored in red.
    - If $\alpha_{e,P} = \alpha_{e,Q}$, then e is colored in green.

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  - If $\alpha_{e,P} = \alpha_{e,Q}$, then e is colored in green.
- Our goal is to show that all edges are green.

Proof

- Let us give weights to vertices according to the following rules. Consider a vertex and any two consecutive edges.
  - If two edges are of the same color, the vertex gets the weight 0.
  - If one of edges is red and another is blue, then the vertex gets the weight 1.
  - If one of the edges is green and another is of different color, then the vertex gets the weight $1/2$.

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- Our first goal is to show that there is a vertex such that its surrounding corners have the total weight at most 3.

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$$\leq 2\sum_{i \geq 3} i f_i - 4\sum_{i \geq 3} f_i = 2 \cdot 2|E| - 4|F|,$$

a contradiction with Euler's formula $|V| - |E| + |F| = 2$.

Local analysis

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- Consider a small sphere S with center at v and look at its intersection with corners (of P and Q). These intersections are spherical polygons. We know that they have the same sides and there is some information about their angles.

Spherical Arm Lemma

Cauchy's arm lemma

Suppose that $q_1 q_2 \dots q_n$ and $q'_1 q'_2 \dots q'_n$ are two spherical convex polygons such that

$$|q_i q_{i+1}| = |q'_i q'_{i+1}|$$

for $i = 1, \dots, n-1$ and

$$\angle q_{j-1} q_j q_{j+1} \leq \angle q'_{j-1} q'_j q'_{j+1}$$

for $j = 2, \dots, n-1$. Then $|q_1 q_n| \leq |q'_1 q'_n|$ with equality iff $\angle q_{j-1} q_j q_{j+1} = \angle q'_{j-1} q'_j q'_{j+1}$ for all $j = 1, \dots, n$.

Plane version of the arm lemma

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Let us use induction on n. For $n = 3$, the statement is trivial because of the cosine theorem:

$$|q_1 q_3|^2 = |q_1 q_2|^2 + |q_2 q_3|^2 - 2|q_1 q_2||q_2 q_3| \cos \angle q_1 q_2 q_3,$$

$$|q'_1 q'_3|^2 = |q'_1 q'_2|^2 + |q'_2 q'_3|^2 - 2|q'_1 q'_2||q'_2 q'_3| \cos \angle q'_1 q'_2 q'_3.$$

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- If there are two equal corresponding angles, say, $\angle q_{i-1}q_iq_{i+1} = \angle q'_{i-1}q'_iq'_{i+1}$, then we can apply induction hypothesis for $q_1 \dots q_{i-1}q_{i+1} \dots q_n$ and $q'_1 \dots q'_{i-1}q'_{i+1} \dots q'_n$. Here we use that $\triangle q_{i-1}q_iq_{i+1} = \triangle q'_{i-1}q'_iq'_{i+1}$.

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- Otherwise, we consider a point q^*_n such that the polygon $q_1 \dots q_{n_1} q^*_n$ is convex, $\angle q_{n-2} q_{n-1} q^*_n \leq \angle q'_{n-2} q'_{n-1} q'_n$ and one of the following two conditions holds:

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  - $\angle q_{n-2}q_{n-1}q^*_n = \angle q'_{n-2}q'_{n-1}q'_n$.
  - The point q^*_n lies on the ray q_2q_1.

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- Otherwise, we consider a point q_n^* such that the polygon $q_1 \dots q_{n_1} q_n^*$ is convex, $\angle q_{n-2} q_{n-1} q_n^* \leq \angle q'_{n-2} q'_{n-1} q'_n$ and one of the following two conditions holds:
  - $\angle q_{n-2} q_{n-1} q_n^* = \angle q'_{n-2} q'_{n-1} q'_n$.
  - The point q_n^* lies on the ray $q_2 q_1$.
- In particular, $\angle q_{n-2} q_{n-1} q_n \leq \angle q_{n-2} q_{n-1} q_n^*$. Therefore, $|q_1 q_n| \leq |q_1 q_n^*|$.
- The first case is based on the induction step explained above. Using it, we have

$$|q_1 q_n| \leq |q_1 q_n^*| \leq |q'_1 q'_n|.$$

- If q_n belongs to the ray $q_2 q_1$, then using the induction step for the polygons $q_2 \dots q_n^*$ and $q_2' \dots q_n'$, we obtain $|q_2 q_n^*| \leq |q_2' q_n'|$.

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$$|q_1'q_n'| \geq |q_2'q_n'| - |q_1'q_2'| \geq |q_2q_n^*| - |q_1q_2| = |q_1q_n|.$$

Plane vs Spherical versions of the arm lemma

The only difference in the argument is the so-called spherical cosine theorem.

Spherical cosine theorem

If ABC be a spherical triangle on a unit sphere, then

$$\cos|AC| = \cos|AB|\cos|BC| + \sin|AB|\sin|BC|\cos\angle ABC.$$

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- Assume that among edges incident to v there are blue and (or) red. Then we easily obtain a contradiction.
- Therefore all edges incident to v are green. Next, we delete vertex v_P and v_Q from P and Q, respectively, and apply induction on the number of vertices in the polytope.