Combinatorial olympiad 2018

Rules.

1. The olympiad is mainly aimed at undergraduate students, but it also open to other participants (including high-school students).
2. We recommend sending solutions in PDF format. Please write your name, email, university and year of university education (if you are a student) on the first page of the document with solutions. The solutions can be sent to comb.olymp@phystech.edu before 01.04.2018.
3. The full solution of each problem will be graded by 10 points, the partial solutions will be also graded.
4. If you have any questions, please email us at comb.olymp@phystech.edu.
5. The results will be available here: http://polyanskii.com/other/combinatorial-olympiad-2018/

The olympiad is organized by the Department of Discrete Mathematics of Moscow Institute of Physics and Technology (State University). Here is information about our very strong international master's program in combinatorics and its applications: https://mipt.ru/education/chairs/dm/master-s-program-in-discrete-mathematics-2018.php. If you have any questions about this program, please email Prof. Andrei Michailovich Raigorodskii at mraigor@yandex.ru.

Problems.

1. An $n \times 100$ matrix in which each element is zero or one is given. The number of zeros in each column of the matrix is greater than the number of ones. Is it true that one can find 3 columns out of 100 columns satisfying the following condition: the number of rows whose intersection with these columns are zeros is greater than the number of rows whose intersection with these columns are ones?

2. a) Prove that if the degree of each vertex of a graph G is d and the diameter of G is 2 then the number of vertices does not exceed $d^2 + 1$.

b) Prove that if this bound is achieved then $d + 1$ is not divisible by 5.

3. Let $p > 2$ be a prime number. Prove that $\mathbb{Z}_p \setminus \{0\}$ can be partitioned into two sets A and B of equal size such that the number of solutions $(a, b) \in A \times B$ of the equation $x = a + b$ is independent of $x \in \mathbb{Z}_p \setminus \{0\}$.

4. Integers $n \geq k \geq r \geq 1$ are given. For a partition of the set $[n] = \{1, \dots, n\}$ into r subsets $A_1, \dots, A_r$, denote by $m_k(A_1, A_2, \dots, A_r)$ the number of k-element subsets of $[n]$ that have non-empty intersection with each A_i, $1 \leq i \leq r$. Describe a partition of $[n]$ into r parts $A_1, \dots, A_r$ delivering the maximum of $m_k(A_1, \dots, A_r)$.

5. There is an infinite sequence of events $A_1, A_2, \dots, A_n, \dots$ of probability 0.1. Prove that there exist distinct $i_1, \dots, i_5$ such that the event $A_{i_1} \cap \dots \cap A_{i_5}$ has probability greater than $0.99 \cdot 10^{-5}$.

6. There are $n \geq 4$ points in general position in the plane, i.e. no 3 of them are collinear. One draws line segments between some pairs of points. It turns out that each segment intersects all other segments but at most one (two segments intersect each other if they share a common point which can possibly be a common endpoint). Find the maximal possible number of segments that can be drawn.

7. Find the maximum constant c_n such that for any vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n+1}$ in $\mathbb{R}^n$ with sum equal to the zero vector there are $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, \dots, \varepsilon_{n+1} = \pm 1$ with the property

$$|\varepsilon_1 \mathbf{v}_1 + \varepsilon_2 \mathbf{v}_2 + \dots + \varepsilon_{n+1} \mathbf{v}_{n+1}|^2 \geq c_n(|\mathbf{v}_1|^2 + \dots + |\mathbf{v}_{n+1}|^2).$$

8. Suppose that a set $K \subset \mathbb{R}^2$ satisfies $K \cap (\mathbf{b} + K) = \varnothing$ for some vector $\mathbf{b}$. Prove that it is impossible to find two paths inside K satisfying the following properties:
1) the first path goes from **a** point $A \in K$ to **a** point $B \in K$ and the second one goes from B to A;
2) two closed discs of diameter $|\mathbf{b}|$ do not intersect while their centers move along the paths (in particular the discs switch their positions).

9. The *clique chromatic number* $\underline{\chi}(G)$ of a graph G without isolated vertices is the least number of colors in a coloring of the graph such that no maximal clique is monochromatic (a clique is maximal if it is not a proper subset of another clique). The graph $G(n, r, s)$ is the graph whose vertex set is the set of all r-element subsets of $[n] = \{1, 2, \dots, n\}$, where two r-element sets A and B are adjacent if and only if $|A \cap B| = s$.
a) Prove that $\underline{\chi}(G(n, r, 0)) = 2$ for $n > N(r)$, where $N(r)$ is some function depending on r.
b) Prove that $\varlimsup\limits_{n\to\infty} \underline{\chi}(G(n, r, s)) = \infty$ for any $r > s > 0$.

10. Let G be a connected non-bipartite k-regular graph on n vertices. The entries of the matrix $A^{(r)} = (a_{ij}^{(r)})_{i,j=1}^n$ are defined in the following way: $a_{ij}^{(r)}$ is the number of non-backtracking paths of length r starting at i and finishing at j, i.e. the number of paths $i = x_0, x_1, \dots, x_r = j$, where x_{i-1} and x_i are adjacent vertices in G for $1 \leq i \leq r$, and $x_{i-1} \neq x_{i+1}$ for any $1 \leq i \leq r - 1$. Find $\lim\limits_{r\to\infty} \frac{\mathrm{tr} A^{(r)}}{(k-1)^r}$.