Rules.

1. The olympiad is mainly aimed at undergraduate students, but it also open to other participants (including high-school students).
2. We recommend sending solutions in PDF format. Please write your name, email, university and year of university education (if you are a student) on the first page of the document with solutions. The solutions can be sent to comb.olymp@phystech.edu before 15.05.2019.
3. The full solution of each problem will be graded by 10 points, the partial solutions will be also graded.
4. If you have any questions, please email us at comb.olymp@phystech.edu.
5. The results will be available here: http://polyanskii.com/other/combinatorial-olympiad-2019/

The olympiad is organized by the Department of Discrete Mathematics of Moscow Institute of Physics and Technology (State University). Here is information about our international master's programs and other opportunities:

- Advanced Combinatorics: https://advcombi.org/
- Contemporary Combinatorics: https://comb-mipt.ru/
- Computer Science: http://cs-mipt.ru/.
- Deep learning school: https://www.dlschool.org/?lang=en

If you have any questions about programs, please email Prof. Andrei Michailovich Raigorodskii at mraigor@yandex.ru.

1. The number 12345678987654321 is written on the blackboard. Alice and Bob play the following game, taking turns. At every turn, each player decreases by 1 or 2 any digit other than the leftmost digit, if the sequence of symbols on the board after the change is a positive integer. A player losses if he cannot make a turn. Who has a winning strategy if Alice starts?

2. There are 1001 rectangles with integer sides such that for each rectangle the difference between the length of sides is less than 100. Prove that there are 21 rectangles such that each of them can cover or be covered by any other (one can rotate rectangles).

3. Lets us call by *compression* the following operation on a graph: simultaneously connect all pairs of vertices at the distance being a power of 2. The *characteristic* of a connected graph is the minimal number of compressions needed to transform the graph to a complete one. Find the maximal *characteristic* of a connected graph with 2019 vertexes.

4. Denote by P and D the perimeter and the diameter of a convex polygon, respectively. Suppose we are given a polyline passing through all the vertices of the polygon. Prove that its length is at least $P - D$.

5. A chess tournament with n players, numbered from 1 to n, was held (so there were $\frac{n(n-1)}{2}$ games in total). The score awarded for each game is 1 for the winner, 0 for the loser, or $\frac{1}{2}$ for both players if there was a draw. It turned out that in all non-draw games winner's number was less than loser's number. Call a game *interesting* if its winner scored less (in total) than its loser.

a. Could it happen that more than a half of games in such a tournament were interesting?

b. Could it happen that more than 99% of games in such a tournament were interesting?

6. Suppose we are given a finite set S of vectors in space, no two of them being collinear. Suppose also that for any two vectors from S there is a third vector from S orthogonal to the both. Prove that one can discard one vector from S so that the rest lies in a plane.

7. Let $A_1, A_2, \dots, A_n$ be pairwise independent random events of probability 1/2 each, where $n \geq 3$. Find the greatest possible probability of their intersection $A_1 A_2 \dots A_n$.

8. Let $f(n)$ denote the number of non-negative integers that can be represented as the sum of a square and a non-negative cube, both less than n; that is, $f(n) = |\{a^2 + b^3 \mid a^2 < n, 0 \leq b^3 < n\}|$. Prove that $f(n) \sim n^{5/6}$, that is, $\lim_{n\to\infty} \frac{f(n)}{n^{5/6}} = 1$.

9. Given 100 points in the plane, prove that they can be coloured in 10 colours in such a way that the convex hulls of the points of each colour share a common point.

10. An $t \times n$ matrix X, where $t > n$, in which each element is zero or one is such that each column contains exactly $s + 1$ ones. We say that a set A of columns *covers* a column b if for each row the sum of intersections of these row with columns in A is greater than the intersection of these row with b. In X there are no distinct s columns *covering* some another column. Prove that $n \geq (s + 1)^2$.

11. Let p be a prime number. The *torus* T is the set of pairs (i, j) or residues modulo p. A *toric line* is the set of pairs $(i, j) \in T$ satisfying the relation $ai + bj + c = 0$, for some fixed residues a, b, c. To each pair $(i, j) \in T$, assign a variable x_{ij} taking values in the residues modulo p. Assume that at the beginning all the values $x_{ij} = 0$. One is allowed to pick a toric line, and add 1 to all variables corresponding to it. How many different variable combinations $(x_{ij})_{i,j=0}^{p-1}$ can one get?

12. Suppose that a graph, whose vertex degrees are odd, has a Hamiltonian cycle. Prove that it has another Hamiltonian cycle.